Applied Mathematics and Computation 246 (2014) 599–607

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on persistence and extinction of a randomized food-limited logistic population model

Dianli Zhao *, Sanling Yuan

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

ARTICLE INFO

Keywords:
Stochastic logistic model
Food-limited assumption
Global stability
Lyapunov functional

ABSTRACT

This paper addresses the issue of the asymptotic behavior for a non-autonomous randomized food-limited logistic population model. Several sufficient conditions are formulated and proved for p-moment persistence and extinction of the population, as well as in sense of almost sure. Results show that food-limited assumption has an influence on the convergence rate of the solution to the equilibria for the deterministic and stochastic model. Some previously known results are improved. Numerical simulations are provided to support the results.

1. Introduction

A logistic differential equation with stochastic perturbation, or shortly, a stochastic logistic equation, is a system of the form

$$dN(t) = N(t)\left(1 - \frac{N(t)}{K}\right)[rdt + \sigma dB(t)]. \tag{1}$$

It has been studied by many authors as population growth model of a single species (see [1–3]). $N(t)$ denotes the population density at time t, r is called the intrinsic rate of growth and K is the carrying capacity assumed to be positive. $B(t)$ is a standard Brown motion representing the effects induced by the environmental noise on the natural growth. σ is the intensity of the noise. Considering the influence of environment, Golec and Sathananthan [4] assume that the coefficients of the model are time-varying and they investigate the stability of non-autonomous randomized population model

$$dN(t) = N(t)\left(1 - \frac{N(t)}{K}\right)[r(t)dt + \sigma(t)dB(t)], \tag{2}$$

where $r(t)$ and $\sigma(t)$ are both continuous functions on $[0, +\infty)$. By assuming that $r(t) \equiv r, \sigma(t) \equiv \sigma$ and $0 < N_0 < K$, known results presented in [1] and [4] are as follows.

(A1) If $r < -\frac{1}{2}\sigma^2$, then $\lim_{t\to\infty} N(t) = K$, a.s. (almost surely);

(A2) If $r > \frac{1}{2}\sigma^2$, then $\lim_{t\to\infty} N(t) = 0$, a.s.

* Corresponding author.
E-mail address: Tc_zhaodianli@139.com (D. Zhao).

http://dx.doi.org/10.1016/j.amc.2014.08.070
0096-3003/

In [5,6], the Allee effect is considered on solution of the stochastic population model.

The determined version of (1) is like $\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right)$. It is violated for nearly all populations, e.g. for a food-limited population. An experiment analysis done by Smith [7] shows the classical logistic model does not fit experimental data very well, and suggested a modification of the logistic equation: $\frac{dN(t)}{dt} = rN(t)\frac{K-N(t)}{K+CN(t)}$ where $C = \frac{rate\ of\ increase\ with\ unlimited\ food}{rate\ of\ replacement\ of\ mass\ in\ thesaturation}$ is a nonnegative constant denoting the delayed effects of the food-limited on the growth of the population. From then on, a lot of authors consider the effect of the food-limited on modified logistic model. The existence, uniqueness, and global stability of positive solutions, periodic solutions and Hopf bifurcation of the deterministic food-limited logistic equation have been investigated (see, for example, [8–12]). In [13], Jiang et al. study a randomized logistic equations with food-limited effect of the form

$$dN(t) = N(t)\frac{K - N(t)}{K + CN(t)}[rdt + \sigma dB(t)], \tag{3}$$

where $C, r, \sigma \in [0, \infty)$ are both constants. They get the following result.

Theorem A. *Let $N(t)$ be a continuous positive solution to Eq. (3) for any initial value $N(0) = N_0$ with $0 < N_0 < K$. If $r > \sigma^2$, then $\lim_{t\to\infty} E\left[(N(t) - K)^2\right] = 0$.*

Here, $E(X)$ denotes the expectation of X. In the sequel, $L^1[0,\infty)$ denotes the family of all the continuous and integrable functions on $[0,+\infty)$. $Z^+ = \{1,2,3,\ldots\}$.

By comparing the above results, we conjecture that $\lim_{t\to\infty} E\left[(N(t) - K)^2\right] = 0$ may be deduced from the condition $r > \frac{1}{2}\sigma^2$, and the food-limited assumption may affect the condition for stability of the equilibrium. Almost sure persistence and extinction is also important problem for study of the population modeled by (3), but no results are found by the authors up to now. In reality, due to the influence of environmental changes, such as changes in weather, habitat destruction and exploitation, the expanding food surplus and other factors, the growth rate and the noise intensity in model (3) may be time-independent (see, for example, [4,14–16]). An interesting problem is whether the time-varying coefficients assumption can be useful for obtaining some results different from that in [13]. Motivated by the above, in this paper, the authors will mainly focus on finding the conditions for the persistence and extinction of population modeled by the non-autonomous randomized food-limited population equation

$$dN(t) = N(t)\frac{K - N(t)}{K + CN(t)}[r(t)dt + \sigma(t)dB(t)], \tag{4}$$

where $r(t)$ and $\sigma(t)$ are both bounded and continuous functions on $[0, +\infty)$.

Note that (4) has two equilibria: 0 and K.

1.1. Deterministic food-limited logistic population model

Before statement of the main results, we list the properties of the deterministic logistic model with food-limited assumption for comparison later. Consider

$$\frac{dN(t)}{dt} = r(t)N(t)\frac{K - N(t)}{K + CN(t)}. \tag{5}$$

It's clear that 0 and K are also the equilibria of the deterministic logistic model (5). The following is mainly about the conditions for convergence of the solution to the equilibrium.

Theorem 1.1. *Let $N(t)$ be a solution of Eq. (5) with $N(0) = N_0 > 0$. If $\lim_{t\to\infty} \int_0^t r(s)ds = \infty$, then $\lim_{t\to\infty} N(t) = K$.*

Proof. Let $N(t)$ be a solution of (5). It is obvious if there is $t_0 \geqslant 0$ such that $N(t_0) = K$, then $N(t) = K$ for all $t > t_0$. Note that $N(t) < K$ implies that $\frac{dN(t)}{dt} \geqslant 0$, and $N(t) > K$ implies that $\frac{dN(t)}{dt} \leqslant 0$ for $N_0 > 0$, if $N(t) \neq K$ for all $t > 0$, the solution of (5) can be presented as

$$\frac{N(t)}{|K - N(t)|^{1+C}} = \frac{N_0}{|K - N_0|^{1+C}} e^{\int_0^t r(s)ds}.$$

Therefore, if $\lim_{t\to\infty} \int_0^t r(s)ds = \infty$, then $\lim_{t\to\infty} N(t) = K$. The proof is complete. □

Corollary 1.2. *Let $N(t)$ be a solution of Eq. (5) with $N(0) = N_0 \in (0, K)$, then $0 < N(t) < K$ for all $t \in R^+$ and the following results hold:*

(i) *$\lim_{t\to\infty} N(t) = 0$ if and only if $\lim_{t\to\infty} \int_0^t r(s)ds = -\infty$ holds.*
(ii) *$\lim_{t\to\infty} N(t) = K$ if and only if $\lim_{t\to\infty} \int_0^t r(s)ds = \infty$ holds.*