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Dynamical behaviors of some iterative methods for multiple roots of nonlinear equations

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A R T I C L E   I N F O

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A B S T R A C T

Recently, more and more higher-order iterative methods for finding the multiple roots of nonlinear equations have been presented. Most of them require the information of the multiplicity of roots. In this paper, the conjugacy maps and the extraneous fixed points of some iterative methods for finding the multiple roots are discussed, basins of attractions of them are also given to demonstrate their dynamical behaviors around the multiple roots for several polynomials.

1. Introduction

A root α of multiplicity m of a nonlinear equation $f(x)=0$ means that $f^{(i)}(\alpha)=0, i=0,1,\cdots,m-1$ and $f^{(m)}(\alpha)\neq 0$. Recently, more and more higher order iterative methods have been presented to find the multiple roots of nonlinear equations [1–16]. Most of them require the information of multiplicity m.

It is well known that the classical Newton's method is of order two for simple roots (i.e. $m=1$), while only linearly converges to the multiple ones ($m\geqslant 2$). However, using the information of multiplicity m, the modified Newton's method has second order of convergence, reads as [17]

NM:

$$x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

In the case of multiple roots with multiplicity m, Traub [18] suggested to use any method for $f^{\frac{1}{m}}$. The modified Newton's method (1) can be viewed as an outstanding example. Some other transformations can be found in [19–23].

Generally, there are two efficient ways to construct higher order iterative methods for multiple roots. One is using the information of the second, even higher order derivative of function f. For example, the cubically convergent Euler–Chebyshev's method [18], Osada's method [5], two equivalent iterative family Laguerre [24] and Hansen–Patrick [25], and etc.

Neta et al. [26] have found that Halley's method [1], a special case of Laguerre family, is one of the best. But after comparing it to Euler–Cauchy's method, another special case of Laguerre family, they realized that the latter is even better [27]. Euler–Cauchy's method is given by

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ECM:

$$x_{n+1} = x_n - \frac{2m}{1 + \sqrt{(2m-1) - 2m\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}\frac{f(x_n)}{f'(x_n)}. \tag{2}$$

Although the second derivative of f is used, the convergence orders of these third-order methods are not optimal order according to the conjecture of Kung and Traub [28]. To our best knowledge, there have been no optimal iterative methods using higher order derivative yet. Moreover, the obvious disadvantage is that it is hard to be evaluated in practical application.

So, another way, using the technique of multi-step, has received more concerns [29–31]. Recently, more and more iterative methods of this type have been presented [2–16]. Some of them are of optimal order. For example, in [14], we construct a more general iteration scheme for multiple roots of optimal order four

$$\begin{cases} y_n & = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = x_n - Q\left(\frac{f'(y_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{3}$$

where the function $Q(\cdot) \in C^2(\mathbb{R})$ and satisfies that

$$Q(u) = m, \quad Q'(u) = -\frac{1}{4}m^3\left(\frac{m+2}{m}\right)^m, \quad Q''(u) = \frac{1}{4}m^4\left(\frac{m+2}{m}\right)^{2m},$$

with $u = (\frac{m}{m+2})^{m-1}$. Family (3) contains almost all optimal fourth-order methods for multiple roots known already [9,10,12]. For example, Li et al. [10] have developed the following special case of family (3)

LCN:

$$\begin{cases} y_n & = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = x_n - (m - \frac{m^2}{2})\frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)}{-\frac{1}{m}f'(x_n) + \frac{1}{m(\frac{m}{m+2})^m}f'(y_n)}, \end{cases} \tag{4}$$

In [26], Neta et al. have compared the dynamical behaviors of some iterative methods for multiple roots, and concluded that **LCN** is the best among them.

Recently, we also presented the following two optimal families [15,16]

$$\begin{cases} y_n & = x_n - m\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = y_n - G(v)\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{5}$$

where $v = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, and the function $G(\cdot)$ satisfies $G(0) = 0,\ G'(0) = m,\ G''(0) = 4m$ and $G'''(0) < +\infty$.

$$\begin{cases} y_n & = x_n - m\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = y_n - G(w_n)\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{6}$$

where $w_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$, and the function $G(\cdot)$ satisfies $G(0) = 0, G'(0) = m, G''(0) = 4m^2/(m-1)$ and $G'''(0) < +\infty$.

These two families are the only ones known to the authors where the root of the function is required at each step. Very recently, Neta et al. give timing comparison for two special cases of family (6) with some other iterative methods [32]. It is shown that one of these two special cases need more running time than others. However, they think that the running time mainly is depended on the convergence rate rather than the evaluation of $(m-1)^{st}$ root.

The aim of this paper is to compare several algorithms from the view point of dynamical behavior [26,33–35]. We shall discuss the conjugacy maps for the polynomial $((z-a)(z-b))^m$ and the extraneous fixed points for $(z^2-1)^m$. We also investigate the comparison in the complex plane using basins of attraction.

The compared methods include **NM**, **ECM**, **LCN**, three special cases of family (3).

ZCS1:

$$\begin{cases} y_n & = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = x_n - \frac{1}{8}m\left(m^3\left(\frac{m+2}{m}\right)^{2m}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2 - 2m^2(m+3)\left(\frac{m+2}{m}\right)^m\frac{f'(y_n)}{f'(x_n)} + (m^3+6m^2+8m+8)\right)\frac{f(x_n)}{f'(x_n)}. \end{cases} \tag{7}$$

ZCS2:

$$\begin{cases} y_n & = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} & = x_n - \left(\frac{1}{8}m^4\left(\frac{m+2}{m}\right)^m\frac{f'(y_n)}{f'(x_n)} + \frac{1}{8}m(m+2)^3\left(\frac{m}{m+2}\right)^m\frac{f'(x_n)}{f'(y_n)} - \frac{1}{4}m(m^3+3m^2+2m-4)\right)\frac{f(x_n)}{f'(x_n)}. \end{cases} \tag{8}$$