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Affine transformational HDMR and linearised rational least squares approximation

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ABSTRACT

High dimensional model representation (HDMR) is a technique that is used to approximate multivariate functions with functions of less number of variables. In transformational high dimensional model representation (THDMR), the HDMR of a transformation of a given multivariate function f can be truncated at the constant term and the inverse transformation of this constant is used as an approximation to this given function. If the transformation is affine having polynomials as coefficients, then the obtained approximation to such f is a rational function.

Since the computation of the best rational approximant of a function is a highly non-linear optimisation problem, the scientists have focused on linearising such minimisation problem and solve it via basic linear algebra tools. The problem of finding polynomials minimising the continuous 2-norm (L_2-norm) of a weighted residual function is called a linearised rational least squares approximation.

The major contribution of this paper is to recognise that if an affine transformation is used in transformational HDMR with a constant approximation, then this independently developed technique coincides with linearised rational least squares approximation.

1. Introduction

Approximations are used in various fields of science as well as in many industrial and commercial fields. The functions appearing in these fields often comprise more than one variable and due to high dimensionality certain difficulties may arise [1]. *High dimensional model representation* (HDMR) is a representation method that aims to eliminate difficulties occurring while dealing with multivariate functions. In HDMR, a square integrable, d-variate function f that is continuous on a rectangular domain in $\mathbb{R}^d$, is expanded as the sum of less variate functions that can be obtained uniquely via multidimensional integrals [2]. f is then approximated with the truncated HDMR.

Transformational high dimensional model representation (THDMR) is a variant of HDMR where, instead of the original function f, a suitable transformation of it is expanded into a sum of less variate functions [3]. Once the HDMR expansion of the transformed function is constructed, an approximant is obtained by truncating this expansion. The inverse transformation of this truncation is the THDMR approximation to the original d-variate function.

If the multivariate function f under consideration is highly non-linear, it can be approximated by a certain rational function whose numerator and denominator are linear combinations of some basis functions so to minimise the error norm. If the function is continuous and square integrable on a bounded region $\Omega \subset \mathbb{R}^d$ and if the continuous 2-norm (L_2-norm) is used

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to measure the error (see [4]), then the technique is called *rational least squares approximation*. Again this minimisation problem is difficult to solve and usually the corresponding linearised problem is considered.

Although THDMR and rational least squares approximation seem quite different at the first glance, in this work it is shown that they coincide under certain conditions. In Section 2, the rational least squares problem and its linearised version are introduced. In Section 3, THDMR results will be given for an affine transformation. We conclude the paper with univariate and bivariate examples, where the quality of the linearised rational least squares approximation is illustrated.

2. Linearised rational least squares approximation

Before introducing linearised rational least squares, we provide the statements of the problem and the existing results in the literature.

2.1. Least squares approximation

Let P_m and Q_n be $(m+1)$ and $(n+1)$ dimensional subspaces of continuous functions on Ω that are spanned by the first $(m+1)$ and $(n+1)$ functions in $\{\phi_i(\mathbf{x})\}_{i=0}^{s}$, where ϕ_i are linearly independent and continuous functions on Ω, $\mathbf{x}=(x_1,\ldots,x_d)$, $s=\max(m,n)$ and Ω is a bounded subset of $\mathbb{R}^d$. We define the following set

$$R_{m,n}=\{r(\mathbf{x})\equiv p(\mathbf{x})/q(\mathbf{x}) : p\in P_m,\ q\in Q_n\}.$$

For a given function f, continuous on Ω, a function $p^*\in P_m$ for which $\|f-p^*\|_{\mathrm{p}}\leqslant\|f-p\|_{\mathrm{p}},\forall p\in P_m$ is called *best linear approximation to f* and a function $r^*\in R_{m,n}$ for which $\|f-r^*\|_{\mathrm{p}}\leqslant\|f-r\|_{\mathrm{p}},\forall r\in R_{m,n}$ is called *best generalised rational approximation to f*. Here $\|\,.\,\|_{\mathrm{p}}$, $1\leqslant \mathrm{p}\leqslant\infty$ is any continuous p-norm.

The existence of best linear approximations is guaranteed for any p-norm, since the set P_m is a finite dimensional subspace of the space of continuous functions equipped with any continuous p-norm [5]. If $1<\mathrm{p}<\infty$ the uniqueness of the best linear approximation is guaranteed for any set of $\{\phi_i(\mathbf{x})\}_{i=0}^{m}$, and if $\mathrm{p}=1$ or $\mathrm{p}=\infty$ the best linear approximation is unique only if $\{\phi_i(\mathbf{x})\}_{i=0}^{m}$ forms a Chebyshev system [6].

In order to guarantee the existence of best generalised rational approximations, some extra conditions need to be imposed on $R_{m,n}$ (see [5]). Some results are given in [4,7] for $d=1$. In the case $d=1,\mathrm{p}=\infty$, supposing that $\{\phi_i(\mathbf{x})\}_{i=0}^{s}$ are a triangular family of polynomials, and that a best rational approximant of f exists then, up to a constant, it is unique. For different norms and different basis functions, there is not much information in the literature on the uniqueness of best generalised rational approximations.

We assume that $f\in L_{2,w}(\Omega)$, where $L_{2,w}(\Omega)$ is the space of continuos, square integrable functions on Ω with respect to the weight function w. The inner product and the norm in $L_{2,w}(\Omega)$ are defined as

$$\langle f,g\rangle=\int_\Omega f(\mathbf{x})g(\mathbf{x})w(\mathbf{x})d\mathbf{x},$$
$$\|f\|_2=\langle f,f\rangle^{\frac{1}{2}}=\left(\int_\Omega f(\mathbf{x})^2 w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{2}}, \tag{1}$$

where w is a non-negative weight function with

$$\int_\Omega w(\mathbf{x})d\mathbf{x}<\infty.$$

Let P_m and Q_n be the linear spaces of d-variate polynomials of total degree at most M and N respectively and $r=p/q$ with

$$p(\mathbf{x})=\sum_{i=0}^{m}a_iu_i(\mathbf{x}),\quad q(\mathbf{x})=\sum_{j=0}^{n}b_ju_j(\mathbf{x}). \tag{2}$$

Here $\{u_i,\ i=0,\ldots,m\}$ and $\{u_j,\ j=0,\ldots,n\}$ are bases for P_m and Q_n, where $m+1=\binom{M+d}{M}$, $n+1=\binom{N+d}{N}$ and

$$\langle u_i,u_j\rangle=\int_\Omega u_i(\mathbf{x})u_j(\mathbf{x})w(\mathbf{x})d\mathbf{x}=\delta_{ij},\quad i,j=0,1,\ldots,s, \tag{3}$$

Let $u_0=1$ and w be a normalised weight function, which implies that

$$\int_\Omega w(\mathbf{x})d\mathbf{x}=1.$$

The problem of finding coefficients $a_i,\ i=0,\ldots,m$ and $b_j,\ j=0,\ldots,n$ in (2) that minimise $\|f-r\|_2$ is a highly non-linear problem.

Instead of finding coefficients that minimise $\|f-r\|_2$, one can solve the corresponding linearised problem i.e. finding p and q such that $\|fq-p\|_2$ is minimised. We stress that the polynomials $p^{(0)}$ and $q^{(0)}$, which minimise $\|f-\frac{p}{q}\|_2$, are not necessarily the polynomials $p^{(1)}$ and $q^{(1)}$ which minimise $\|fq-p\|_2$. We will illustrate this with the following example.