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Boundedness solutions of the complex Lorenz chaotic system

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A R T I C L E I N F O

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A B S T R A C T

This paper is concerned with the boundedness of solutions of the complex Lorenz system. We have obtained the global exponential attractive set $\Psi_{\lambda,m}$ and the ultimate bound $\Omega_{\lambda,m}$ for this system. Furthermore, we confirm that the rate of the trajectories of the system going from the exterior of the set $\Psi_{\lambda,m}$ to the interior of the set $\Psi_{\lambda,m}$ is an exponential rate. The rate of the trajectories is also obtained. Numerical simulations are presented to show the effectiveness of the proposed scheme.

1. Introduction

To estimate the boundedness of a chaotic system is a challenging but an interesting work in general [1–14]. A chaotic system is bounded, in the sense that its phase portraits are bounded in the phase space. And the ultimate boundedness of a chaotic system plays an important role in chaos control, chaos synchronization, and many other applications. If we can show that a chaotic or a hyperchaotic system has a global attractive set, then the system cannot possess hidden attractors outside the global attractive set. This is very important for engineering applications, since it is very difficult to predict the existence of hidden attractors and they can lead to crashes [15–17]. The boundedness of the Lorenz system were first studied by G.A. Leonov [18]. Then, the ultimate boundedness of other chaotic systems, including the synchronous motor system [19], a new chaotic system [20], the hyperchaotic Lorenz–Haken system [21], the Lü system [22] and the generalized Lorenz chaotic systems [23], was also studied and some important results were obtained. However, it is a very difficult task to estimate the boundedness of the chaotic systems [22,23]. The construction of new Lyapunov functions is always a piece of art, since there is no regular way to find one. Therefore, it is necessary to study the boundedness of the complex Lorenz chaotic system.

Since Fowler et al. introduced the complex Lorenz equations [24], many complex chaotic systems have been proposed and studied in the last few decades. For example, Mahmoud et al. introduced the complex Chen and Lü systems [25]. It is well known that the complex chaotic systems have more widely applying space [25]. Such as secure communication, synchronization, control, etc. [25]. Another interesting application that discovered was the anti-synchronization, which has been investigated both experimentally and theoretically in many physical systems [26,27].

2. Mathematical model

State equations of the complex Lorenz chaotic system can be described as follows [28]:

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$$\begin{cases} \dot{y}_1 = a(y_2 - y_1), \\ \dot{y}_2 = by_1 - y_2 - y_1 y_3, \\ \dot{y}_3 = \frac{1}{2}(y_1\bar{y}_2 + \bar{y}_1 y_2) - cy_3, \end{cases} \tag{1}$$

where $y_1 = u_1 + ju_2,\ y_2 = u_3 + ju_4,\ y_3 = u_5,\ j = \sqrt{-1}, \bar{y}_1$ and $\bar{y}_2$ are conjugate complex numbers of y_1 and y_2. Replacing complex variables in system (1) with real number variables and imaginary number variables, Zhang et al. get an equivalent system as follows (see [28]):

$$\begin{cases} \dot{u}_1 = a(u_3 - u_1), \\ \dot{u}_2 = a(u_4 - u_2), \\ \dot{u}_3 = bu_1 - u_3 - u_1 u_5, \\ \dot{u}_4 = bu_2 - u_4 - u_2 u_5, \\ \dot{u}_5 = u_1 u_3 + u_2 u_4 - cu_5, \end{cases} \tag{2}$$

where a, b, c are positive parameters of system (2). When $a = 35,\ b = 55,\ c = \frac{8}{3}$, the system (2) is chaotic [28]. Phase portraits of system (2) are shown in Figs. 1 and 2.

Remark. While positive Lyapunov exponent is widely used as indication of chaos, rigorous consideration requires verification of additional properties of considered system (such as regularity, ergodicity), because of so-called Perron effects of Lyapunov exponents sign reversal (see excellent papers [29–31] for a detailed discussion of the attractor).

Some basic dynamical properties of the complex Lorenz system (2) were studied in [28]. But many properties of the complex Lorenz remains unknown. In the following, we will discuss the boundedness of the complex Lorenz system (2).

The following structure of this paper is organized as follows: In Section 3, we will study the ultimate boundedness of system (2). In Section 4, we will study the global exponential attractive set of system (2). Conclusion remarks will be given in Section 5.

3. The ultimate boundedness

In this section, we will discuss the boundedness of the complex Lorenz system (2) for any $a > 0$, $b > 0$, $c > 0$. Before going into details, let us introduce the following lemma.

Lemma 1. *Define*

$$\Pi_1 = \left\{ (x_1, x_2, y_1, y_2, z) \middle| \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{(z-c)^2}{c^2} + \frac{y_1^2}{d^2} + \frac{y_2^2}{e^2} = 1,\ a \neq 0,\ b \neq 0,\ c \neq 0,\ d \neq 0,\ e \neq 0 \right\}, \tag{3}$$

and

$$G_1(x_1, x_2, y_1, y_2, z) = x_1^2 + x_2^2 + y_1^2 + y_2^2 + z^2, \quad H_1(x_1, x_2, y_1, y_2, z) = x_1^2 + x_2^2 + y_1^2 + y_2^2 + (z - 2c)^2, \quad (x_1, x_2, y_1, y_2, z) \in \Pi_1.$$

Then, we have the conclusions that

$$\max_{(x_1,x_2,y_1,y_2,z)\in\Pi_1} G_1 = \max_{(x_1,x_2,y_1,y_2,z)\in\Pi_1} H_1 = \begin{cases} \frac{a^4}{a^2-c^2}, & a \geqslant b,\ a \geqslant d,\ a \geqslant e,\ a \geqslant \sqrt{2}c, \\ \frac{b^4}{b^2-c^2}, & b > a,\ b \geqslant d,\ b > e,\ b \geqslant \sqrt{2}c, \\ \frac{d^4}{d^2-c^2}, & d > a,\ d > b,\ d \geqslant e,\ d \geqslant \sqrt{2}c, \\ \frac{e^4}{e^2-c^2}, & e > a,\ e \geqslant b,\ e > d,\ e \geqslant \sqrt{2}c, \\ 4c^2, & a < \sqrt{2}c,\ b < \sqrt{2}c,\ d < \sqrt{2}c,\ e < \sqrt{2}c. \end{cases}$$

Proof. It can be easily proved by the Lagrange multiplier method. □

Lemma 2. *Define a set*

$$\Gamma_0 = \left\{ (x, y, z, w) \middle| \frac{x^2}{\tilde{a}^2} + \frac{y^2}{\tilde{b}^2} + \frac{(z-\tilde{c})^2}{\tilde{c}^2} + \frac{w^2}{\tilde{d}^2} = 1,\quad \tilde{a} \neq 0,\ \tilde{b} \neq 0,\ \tilde{c} \neq 0,\ \tilde{d} \neq 0 \right\}, \tag{4}$$

and

$$G(x, y, z, w) = x^2 + y^2 + z^2 + w^2, \quad H(x, y, z, w) = x^2 + y^2 + (z - 2\tilde{c})^2 + w^2, \quad (x, y, z, w) \in \Gamma_0.$$