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Some new improved classes of convergence towards Euler's constant

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ABSTRACT

In this paper, using continued fractions, some new quicker sequences convergent to Euler's constant are provided. Finally, for demonstrating the superiority of our new convergent sequence over DeTemple's sequence, Vernescu's sequence and Mortici's sequences, some numerical computations are also given.

1. Introduction

In the theory of mathematical constants, an important concern is the definition of new sequences which converge to these fundamental constants with increasingly higher speed. These convergent sequences and constants play a key role in many areas of mathematics and science in general, as theory of probability, applied statistics, physics, special functions, number theory, or analysis.

One of the most useful sequence in mathematics is

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{1.1}$$

which converge towards the well-known Euler's constant

$$\gamma = 0.577215\ldots.$$

Up until now, many researchers made great efforts in the area of concerning the rate of convergence of the sequence $(\gamma_n)_{n\geqslant 1}$ and establishing faster sequences that converge to Euler's constant and had a lot of inspiring results. For example, in [4,5,14,15], the following estimates are established

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.2}$$

using interesting geometric interpretations. In [13], Vernescu provided the sequence

$$V_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \tag{1.3}$$

for which

$$\frac{1}{12(n+1)^2} < \gamma - V_n < \frac{1}{12n^2}. \tag{1.4}$$

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In [1,2], DeTemple introduced a faster convergent sequence $(R_n)_{n\geqslant 1}$ to γ as follows,

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right), \tag{1.5}$$

which decreases to γ with the rate of convergence n^{-2}, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{1.6}$$

Both (1.3) and (1.5) are slight modifications of Euler's sequences (1.1), but improve the rate of convergence from n^{-1} to n^{-2}.

Recently, Mortici researched Euler's constant again, and provided some convergent sequences which are faster than (1.1), (1.3) and (1.5).

In [6], Mortici provided the following two sequences

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{(6-2\sqrt{6})n} - \ln\left(n + \frac{1}{\sqrt{6}}\right) \tag{1.7}$$

and

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{(6+2\sqrt{6})n} - \ln\left(n - \frac{1}{\sqrt{6}}\right). \tag{1.8}$$

Both sequences (1.7) and (1.8) were shown to converge to γ as n^{-3}.

Next, in [8], Mortici introduced the following class of sequences of the form

$$\mu_n(a,b) = \sum_{k=1}^{n} \frac{1}{k} + \ln\left(e^{a/(n+b)} - 1\right) - \ln a, \tag{1.9}$$

where a, b are real parameters, $a > 0$. Furthermore, he proved that among the sequences $(\mu_n(a,b))_{n\geqslant 1}$, the privileged one

$$\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right)$$

offers the best approximations of γ, since

$$\lim_{n\to\infty} n^3\left(\mu_n\left(\frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4}\right) - \gamma\right) = \frac{\sqrt{2}}{96}. \tag{1.10}$$

Following their work, recently, in [3], we used continued fraction approximation to provide a new quicker sequence convergent to Euler's constant as follows,

$$r_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}}, \tag{1.11}$$

where $a_1 = \frac{1}{2}, a_2 = \frac{1}{6}, a_3 = -\frac{1}{6}, a_4 = \frac{3}{5}, \ldots$.

In this paper, using the same idea from the well-known sequence γ_n to Vernescu's sequence (1.3) and Mortici's sequence 1.7 and 1.8, based on the early works of Mortici and DeTemple, we provide some more general continued fraction approximation than (1.11) for Euler's constant as follows,

Theorem 1.1. *For $r \in (0,\infty)$, we have the following convergent sequence for Euler's constant, if $r \neq 2$,*

$$L_{r,n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{rn} - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}}, \tag{1.12}$$

where

$$a_1 = \frac{2-r}{2r}, a_2 = \frac{r}{6(2-r)}, a_3 = \frac{r}{6(r-2)}, a_4 = \frac{3(2-r)}{5r}, \ldots.$$

If $r = 2$, we have

$$L_{2,n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n - \frac{1}{n}\cfrac{b_1}{n + \cfrac{b_2}{n + \cfrac{b_3}{n + \cfrac{b_4}{n + \ddots}}}}, \tag{1.13}$$