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Solutions of elliptic integrals and generalizations by means of Bessel functions

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A B S T R A C T

The elliptic integrals and its generalizations are applied to solve problems in various areas of science. This study aims to demonstrate a new method for the calculation of integrals through Bessel functions. We present solutions for classes of elliptic integrals and generalizations, the latter, refers to the hyperelliptic integrals and the integral called Epstein–Hubbell. The solutions obtained are expressed in terms of power series and/or trigonometric series; under a particular perspective, the final form of a class of hyperelliptic integrals is presented in terms of Lauricella functions. The proposed method allowed to obtain solutions in ways not yet found in the literature.

1. Introduction

The elliptic integrals can be viewed as generalizations of inverse trigonometric functions and can provide solutions to various problems in Electromagnetism [1,2], Fluid Mechanics [3] and Chemical Reactions Kinetics [4]. In particular, in geometry are used to measure the perimeter of an ellipse and the area of an ellipsoid [5,6].

In general, those integrals cannot be expressed in terms of elementary functions (all attempts failed, until the nineteenth century when it was finally proved, in fact, to be impossible to carry out this objective [7]).

The solutions obtained in this work are compared with the usual forms of the literature, which are in most cases, different in the representation forms. In the present work, the method used to solve the elliptic integrals is to relate them to the Bessel functions.

Under a particular perspective, the complete elliptic integrals are expressed in terms of power series, while the incomplete forms are expressed in terms of power and trigonometric series. The complete elliptic integral of the first kind obtained with the proposed method is the same form found in the literature [8,9]. For the complete elliptic integrals of the second and third kind, the solutions are given in a different form from the usual representations [10,11,8]. The incomplete elliptic integrals of the first, second and third kind are also solved with the Bessel function's method. In exception of the incomplete elliptic integral of the first kind, the other expressions are not found in the literature.

The Epstein–Hubbell integral, called as a generalization of elliptic integrals [9], is of a significant importance and is used in the solving of various problems [12,13]. For this integral, we propose a new solution, through the same method here used in other elliptic integrals.

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Hyperelliptic integrals are also addressed in this work, specifically, two types of these integrals are proposed with their respective solutions through Bessel functions. One of these integrals is expressed in its final form as a series of Lauricella functions.

1.1. Definition of elliptic integrals

An elliptic integral is any function that can be expressed as,

$$\int R(x, \sqrt{P(x)})dx, \tag{1.1}$$

where $P(x)$ is a polynomial of third or fourth degree.

With some reductions, each elliptic integral can be represented in a form involving integrals of rational, trigonometric, inverse trigonometric, logarithmic and exponential functions; Bronshtein et al. [14], Jeffrey and Dai [9].

2. Complete elliptic integral of the first kind

In this work, we propose that the complete elliptic integral of the first kind can be written in terms of a modified Bessel function of the first kind zeroth order, as expressed in the following:

$$F\left(k,\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^\infty I_0(xk\sin\theta)e^{-x}dx \int_0^{\frac{\pi}{2}} d\theta, \tag{2.1}$$

$$F\left(k,\frac{\pi}{2}\right) = \int_0^\infty e^{-x}dx \int_0^{\frac{\pi}{2}} I_0(xk\sin\theta)d\theta. \tag{2.2}$$

The relationship between a Bessel function of the first kind zeroth order and its modified Bessel function is represented as found in Bowman [15],

$$I_v(z) = i^{-v}J_v(iz), \tag{2.3}$$

$$I_0(z) = J_0(iz). \tag{2.4}$$

The following representation found in Abramowitz and Stegun [10] allows to rewrite the integral form of the Bessel function in Eq. (2.2) in a closed form expression:

$$\int_0^{\frac{\pi}{2}} J_n(2iz\sin\theta)d\theta = \frac{\pi}{2}J_n^2(iz), \tag{2.5}$$

$$= \int_0^{\frac{\pi}{2}} I_n(2z\sin\theta)d\theta = \frac{\pi}{2}I_n^2(z). \tag{2.6}$$

After this, replacing $n = 0$ and $z = kx/2$, the integral forms are represented by:

$$\int_0^{\frac{\pi}{2}} J_0(ikx\sin\theta)d\theta = \frac{\pi}{2}J_0^2\left(i\frac{kx}{2}\right), \tag{2.7}$$

$$= \int_0^{\frac{\pi}{2}} I_0(kx\sin\theta)d\theta = \frac{\pi}{2}I_o^2\left(\frac{kx}{2}\right). \tag{2.8}$$

Substituting the Eq. (2.8) in Eq. (2.2.), we have,

$$F\left(k,\frac{\pi}{2}\right) = \int_0^\infty e^{-x}dx \int_0^{\pi/2} I_0(xk\sin\theta)d\theta = \frac{\pi}{2}\int_0^\infty e^{-x}I_0^2\left(\frac{kx}{2}\right)dx. \tag{2.9}$$

The Bessel function in Eq. (2.9) can be represented by a series, obtained from the following expression found in Abramowitz and Stegun [10]:

$$J_v(z)J_\mu(z) = \left(\frac{z}{2}\right)^{v+z}\sum_{r=0}^\infty \frac{(-1)^r\Gamma(v+\mu+2r+1)\left(\frac{1}{4}z^2\right)^r}{\Gamma(v+r+1)\Gamma(\mu+r+1)\Gamma(v+\mu+r+1)r!}. \tag{2.10}$$

After the substitutions: $v = \mu = 0$, we obtain the form,

$$I_0^2(z) = \sum_{r=0}^\infty \frac{(2r)!z^{2r}}{2^{2r}(r!)^4}. \tag{2.11}$$