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Cubic spline wavelets with short support for fourth-order problems

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ABSTRACT

In the paper, we propose a construction of new cubic spline-wavelet bases on the unit cube satisfying homogeneous Dirichlet boundary conditions of the second order. The basis functions have small supports and wavelets have vanishing moments. We show that stiffness matrices arising from discretization of the biharmonic problem using a constructed wavelet basis have uniformly bounded condition numbers and these condition numbers are very small. We present quantitative properties of the constructed bases and we show a superiority of our construction in comparison to some other cubic spline wavelet bases satisfying boundary conditions of the same type.

1. Introduction

In recent years wavelets have been successfully used for solving various types of differential equations [8,9] as well as integral equations [17,19,20]. The quantitative properties of wavelet methods strongly depend on the choice of a wavelet basis, in particular on its condition number. Therefore, a construction of a wavelet basis is an important issue.

In this paper, we propose a construction of cubic spline wavelet bases on the interval that are well-conditioned, adapted to homogeneous Dirichlet boundary conditions of the second order, the wavelets have vanishing moments and the shortest possible support. The wavelet basis of the space $H_0^2\left((0,1)^2\right)$ is then obtained by an isotropic tensor product. We compare the condition numbers of the corresponding stiffness matrices for various constructions. Finally, a quantitative behavior of an adaptive wavelet method for several boundary-adapted cubic spline wavelet bases is studied.

First of all, we summarize the desired properties of a constructed basis:

- *Riesz basis property.* We construct Riesz bases of the space $H_0^2(0,1)$ and $H_0^2\left((0,1)^2\right)$.
- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four.
- *Vanishing moments.* The inner wavelets have two vanishing moments, the wavelets near the boundary can have less vanishing moments.
- *Short support.* The wavelets have the shortest possible support for a given number of vanishing moments.
- *Locality.* The primal basis functions are local.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form.

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– *Homogeneous Dirichlet boundary conditions.* Our wavelet bases satisfy homogeneous Dirichlet boundary conditions of the second order.
– *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Moreover, in a comparison with constructions in [1,4,11,21,22] that are quite long and technical, the construction in this paper is very simple. Many constructions of cubic spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [2,4,11,21] cubic spline wavelets on the interval were constructed. In [10] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of the second order in [22]. In these cases dual functions are known and are local. Cubic spline wavelet or multiwavelet bases where duals are not local were constructed in [7,14–16]. Some of these bases were already adapted to boundary conditions and used for solving differential equations [6,18]. The advantage of our construction is the shortest possible support for a given number of required vanishing moments. Vanishing moments are necessary in some applications such as adaptive wavelet methods [8,9]. Originally, these methods were designed for wavelet bases with local duals. However, it was shown in [12] that wavelet bases without local dual basis can be used if the solved equation is linear.

This paper is organized as follows: In Section 2 we briefly review the concept of wavelet bases. In Section 3 we propose a construction of primal and dual scaling bases. The refinement matrices are computed in Section 4. In Section 5 the properties of the projectors associated with constructed bases are derived and the proof that the bases are indeed Riesz bases is given. Quantitative properties of constructed bases and other known cubic spline wavelet and multiwavelet bases are studied in Section 6. In Section 7 we compare the number of basis functions and the number of iterations needed to resolve the problem with desired accuracy for bases constructed in this paper and bases from [4,22]. A numerical example is presented for an equation with the biharmonic operator in two dimensions.

2. Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. In this paper, we consider the domain $\Omega = (0,1)$ or $\Omega = (0,1)^2$. We denote the Sobolev space or its subspace by $H \subset H^s(\Omega)$ for nonnegative integer s and the corresponding inner product by $\langle \cdot,\cdot \rangle_H$, a norm by $\|\cdot\|_H$ and a seminorm by $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j,k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level*. Let

$$\|\mathbf{v}\|_{l^2(\mathcal{J})} := \sqrt{\sum_{\lambda\in\mathcal{J}} |v_\lambda|^2}, \quad \text{for } \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\ v_\lambda \in \mathbb{R} \tag{1}$$

and

$$l^2(\mathcal{J}) := \left\{\mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\ v_\lambda \in \mathbb{R},\ \|\mathbf{v}\|_{l^2(\mathcal{J})} < \infty\right\}. \tag{2}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *(primal) wavelet basis* of H, if

(i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c,\ C \in (0,\infty)$ such that

$$c\|\mathbf{b}\|_{l^2(\mathcal{J})} \leqslant \left\| \sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leqslant C\|\mathbf{b}\|_{l^2(\mathcal{J})}, \quad \mathbf{b} := \{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^2(\mathcal{J}). \tag{3}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies}(3)\},\ C_\psi := \inf\{C : C \text{ satisfies}(3)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi/c_\psi$ is called the *condition number* of Ψ.

(ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \leqslant C2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \left\{\tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}}\right\} \subset H$ biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j}\delta_{k,l}, \quad \text{for all } (i,k) \in \mathcal{J},\ (j,l) \in \tilde{\mathcal{J}}, \tag{4}$$

where $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. This family is also a Riesz basis for H, but the functions $\tilde{\psi}_{j,l}$ need not be local. The basis $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j\geqslant j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots H \tag{5}$$

and $\cup_{j\geqslant j_0} V_j$ is complete in H.