Applied Mathematics and Computation 243 (2014) 114–120

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Center conditions and limit cycles for a class of nilpotent-*Poincarè* systems ☆

Feng Li *, Yusen Wu

School of Science, Linyi University, Linyi 276005, Shandong, PR China
School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, Henan, PR China

ARTICLE INFO

Keywords:
Nilpotent *Poincarè* systems
Center
Focus
Lyapunov constant
Rotated vector field

ABSTRACT

A class of nilpotent-*Poincarè* system is discussed in this paper. Center conditions are obtained by methods of inverse integrating factor and theory of rotated vector field. When $n > 2$, we proved that there are $2n + 4$ small amplitude limit cycles enclosing the origin $O(0, 0)$. When $n = 2$, there are 14 limit cycles enclosing the origin $O(0, 0)$.

1. Introduction

A famous and difficult problem for ordinary differential systems is how to characterize their focus and centers. The center problem is directly connected through the *Poincarè*–Lyapunov theorem with the integrability problem. Consider a planar analytic differential system in the form of a linear center perturbed by higher order terms, that is,

$$\dot{u} = -v + U(u, v), \quad \dot{v} = u + V(u, v), \tag{1.1}$$

where U and V are real analytic functions whose series expansions in a neighborhood of the origin start in at least second order terms. Taking polar coordinates we can see that near the origin either all non-stationary trajectories of (1.1) are ovals (in which case the origins a center) or they are spirals (in which case the origin is a focus). By the *Poincarè*–Lyapunov theorem, system (1.1) has a center at the origin if and only if there is a first integral

$$\phi(u, v) = u^2 + v^2 + \sum_{k+j=3}^{\infty} \phi_{kj} u^k v^j, \tag{1.2}$$

where the series converge in a neighborhood of the origin. To distinguish between a center and a focus at the origin of system (1.1) is the so-called *center problem*.

Analytic systems having a nilpotent singular point at the origin were studied by Andreev [1] in order to obtain their local phase portraits. However, Andreev's results do not distinguish between a focus and a center. Takens [2] provided a normal form for nilpotent center of foci. Moussu [3] found the C^∞ normal form for analytic nilpotent centers. Berthier and Moussu in [4] studied the reversibility of the nilpotent centers. Teixeria and Yang [5] analysed the relationship between reversibility and the center-focus problem written in a convenient normal form.

☆ This research is partially supported by the National Nature Science Foundation of China (11201211, 11101126) and Nature Science Foundation of Shan-dong Province (ZR2012AL04) and Applied Mathematics Enhancement Program of Linyi University.
* Corresponding author at: School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, Henan, PR China.
E-mail address: lf0539@126.com (F. Li).

http://dx.doi.org/10.1016/j.amc.2014.05.121
0096-3003/

Poincarè systems

$$\frac{dx}{dt} = -y + xH(x,y),\quad \frac{dy}{dt} = x + yH(x,y), \tag{1.3}$$

where $H(x,y)$ is a polynomial have been studied in many papers; see [1–6] and references therein.

When the origin is a three-order nilpotent critical point, system (1.3) could be written as

$$\frac{dx}{dt} = y + xH(x,y),\quad \frac{dy}{dt} = -2x^3 + yH(x,y). \tag{1.4}$$

As far as I know, there is no paper to investigate the center-focus problem for this system. Now, we called system (1.4) to be nilpotent-*Poincarè* system. In this paper, a class of nilpotent-*Poincarè* system

$$\frac{dx}{dt} = y + x(H_4(x,y) + H_{4n}(x,y)),\quad \frac{dy}{dt} = -2x^3 + y(H_4(x,y) + H_{4n}(x,y)) \tag{1.5}$$

will be discussed. The main goal of this paper is to use the integral factor method and theory of rotating vector fields to distinguish center–focus and give the conditions of center.

This paper is divided into three sections. In Section 2, we restate some known results necessary to demonstrate the main results. In Section 3, we obtain the conditions of center for the system.

2. Preliminary knowledge

Some related notions and results could be seen in [7,8]. When the origin of system (1.4) is a three-order monodromic critical point, the system could be written as the following form.

$$\frac{dx}{dt} = y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^iy^j = X(x,y),\quad \frac{dy}{dt} = -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^iy^j = Y(x,y). \tag{2.1}$$

Theorem 2.1. *For any positive integer s and a given number sequence*

$$\{c_{0\beta}\}\quad \beta \geqslant 3, \tag{2.2}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$ of the formal series

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x,y), \tag{2.3}$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty}\omega_m(s,\mu)x^m, \tag{2.4}$$

where for all $k, M_k(x,y)$ is a k-homogeneous polynomial of x,y and $s\mu = 0$.

Theorem 2.2. *For $\alpha \geqslant 1, \alpha+\beta \geqslant 3$ in (2.3) and (2.4), $c_{\alpha\beta}$ can be uniquely determined by the recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \tag{2.5}$$

For $m \geqslant 1, \omega_m(s,\mu)$ can be uniquely determined by the recursive formula

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{2.6}$$

$$\lambda_m = \frac{\omega_{2m+4}(s,\mu)}{2m-4s-1}. \tag{2.7}$$