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A study on vaccination models for a seasonal epidemic process

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ABSTRACT

In this paper seasonal epidemiological processes are considered and a strategy of periodic vaccination is proposed. The invariant formulations associated with an N-periodic system and the reproduction numbers associated with them are considered. A collection of measures to study the stability of the system is introduced. Moreover, the collection of s-basic reproduction number at time j help us to establish conditions on the periodic vaccination rates in the vaccination program. Finally, an SIR model is showed and a comparison between the results obtained using constant or periodic vaccination program is analyzed.

1. Introduction

The mathematical representation of an epidemiological process or a population growth process by means of a continuous-time or a discrete-time dynamic system has long been considered by many researchers, see [1–3]. In the study of these processes plays an important role the concept of basic reproduction number. This concept is defined as the spectral radius of a matrix or a linear operator constructed from the coefficient of the system, see [4,5]. The basic reproduction number is not just a measure or indicator to know whether the disease will be disappear altogether, but it is also a key to establish a threshold vaccination rate necessary to eradicate the disease, see for instance [6,7].

Many epidemiological processes present seasonality. For example, rates of recovered individuals or of infected individuals may change with periodic behavior, according to the external conditions that affect the process. In this case, in the mathematical model appears periodic functions or matrices which hamper the analysis of the problem. The basic reproduction number associated with a model with periodic coefficients of period N has been defined by different authors, see [8–13]. In particular, in [8,9] this concept is introduced as the spectral radius of a matrix which contains all the information for a period N, moreover, in [9] is defined the seasonal type reproductive numbers with the idea of measuring the effort needed to bring the population to extinction by decreasing the births during a particular season and in [13] the authors define N reproduction numbers associated with the system.

The most common vaccination program is to vaccinate all individuals with a constant rate, see [8]. Perhaps it is the most direct method, but it is not the most effective. Some studies have considered specific cases of type SEIR or SEIRS continuous-time and they have done studies with pulse vaccination, see [14,15].

Since it is important to take into account the periodicity of the process in the vaccination program, in this paper we consider seasonal epidemiological processes and we propose a strategy of periodic vaccination. Theory of linear dynamical systems in discrete-time and theory of nonnegative matrices and M- matrices are the support of the theoretical development

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of this study. We introduce new measures or indicators to study the stability of the system. Finally, these new reproduction numbers help us to establish conditions on the periodic vaccination rates.

2. Preliminaries and statement of work

Consider an invariant system

$$x(k+1) = Ex(k), \quad E = T + F, \; k \geqslant 0, \tag{1}$$

where the vector $x(k)$ represents the infected individuals at time k, the nonnegative matrices $T \geqslant O$ and $F \geqslant O$ are the transition matrix and the infection matrix, respectively.

The basic reproduction number, $\mathcal{R}_0$, quantifies the transmission potential of the disease and it gives information on transmissibility and contact rates. This indicator is defined by $\mathcal{R}_0 = \rho(F(I-T)^{-1})$, see [4]. Considering that the initial system without new infections is asymptotically stable $\rho(T) < 1$, it is known (see [16]) that if $\mathcal{R}_0 < 1$ then the system (1) is asymptotically stable, that is, $\rho(E) = \rho(T+F) < 1$. Moreover, if $\mathcal{R}_0 \geqslant 1$, the system (1) is not asymptotically stable $\rho(E) = \rho(T+F) \geqslant 1$.

The basic reproduction number is used to find the threshold value of the vaccination rate, denoted by v, in order to obtain the eradication of the disease. If the matrix F acts on the unvaccinated population then the vaccination model is given by

$$x(k+1) = E^v x(k), \quad E^v = T + F(1-v), \; k \geqslant 0.$$

Now, the new basic reproduction number is given by $\mathcal{R}_0^v = (1-v)\mathcal{R}_0$ and $1 - \frac{1}{\mathcal{R}_0}$ is the threshold value for v such that the model of vaccination becomes asymptotically stable $\rho(E^v) < 1$, see [7]. That is, if $v > 1 - \frac{1}{\mathcal{R}_0}$ then $\mathcal{R}_0^v < 1$.

When this procedure is applied to a seasonality process the difficulty of the problem increase. This seasonality leads to the periodicity of the coefficient matrices of the model. In this case the process is represented by an N-periodic system given by

$$x(k+1) = E(k)x(k), \quad E(k) = T(k) + F(k), \; k \geqslant 0, \tag{2}$$

where the matrices are N-periodic matrices, that is, $T(k+N) = T(k) \geqslant O$, and $F(k+N) = F(k) \geqslant O, \; k \in \mathbb{Z}$.

In the literature, the N-periodic systems have been studied using the following two invariant formulations:

- *N***IS**: A collection of N invariant systems (see [17]), given by

$$x_s(k+1) = E_s x_s(k), \quad E_s = T_s + B_s F_s, \; k \geqslant 0,$$

where

$$x_s(k) = x(kN+s), \quad B_s = row[\Phi_T(s+N, s+1+j)]_{j=0}^{N-1},$$
$$T_s = \Phi_T(s+N, s), \quad F_s = \mathrm{col}[F(s+j)\Phi_E(s+j, s)]_{j=0}^{N-1},$$
$$\Phi_T(s, s_0) = \begin{cases} \prod_{i=s_0}^{s-1} T(s_0 + s - 1 - i), & s > s_0 \\ I_n, & s = s_0 \end{cases}$$

for $s = 0, 1, \ldots, N-1$.

- **ICAS**: An invariant system called *Invariant Cyclically Augmented System*, that recollect all the information of a period (see [18]) which state is defined by $z(k) = M_n^{k-1}\hat{x}(k)$, with $\hat{x}(k) = col[x(k+j)]_{j=0}^{N-1}$, and $M_j = \begin{bmatrix} O & I_{(N-1)j} \\ I_j & O \end{bmatrix}, \; j > 0$.

  This invariant system is given by

$$z(k+1) = E_e z(k), \quad E_e = T_e + F_e, \; k \geqslant 0,$$

where T_e and F_e have the following structure

$$M_e = \begin{pmatrix} O & M(0) \\ \mathrm{diag}(M(1) \; \cdots \; M(N-1)) & O \end{pmatrix},$$

using the corresponding N-periodic collection of matrices.

It is known that the N-periodic system is asymptotically stable if and only if $\rho(E_s) < 1$ or equivalently $\rho(E_e^N) < 1$, since $E_e^N = \mathrm{diag}(E_1 \; E_2 \; \cdots \; E_{n-1} \; E_0)$.

Given an N-periodic system, a first approach of concept of basic reproduction number leads us to define this measure as the basic reproduction number of its ICAS formulation, that is, $\mathcal{R}_0^e = \rho(F_e(I-T_e)^{-1})$, see [8,9,19]. Another way is to define it using its NIS formulation, that is, $\mathcal{R}_0^s = \rho(B_s F_s (I-T_s)^{-1}), \; s = 0, 1, \ldots N-1$, see [13].

In order to get the eradication of the disease we can plan different ways of actuation. If we consider a constant vaccination program, that is, if we consider in all seasons the infection periodic matrix at time k equal to $(1-v)F(k)$, the vaccination model is given by

$$x(k+1) = E^v(k)x(k), \quad E^v(k) = T(k) + (1-v)F(k), \; k \geqslant 0.$$