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Approximate controllability of fractional delay dynamic inclusions with nonlocal control conditions

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ABSTRACT

We introduce a nonlocal control condition and the notion of approximate controllability for fractional order quasilinear control inclusions. Approximate controllability of a fractional control nonlocal delay quasilinear functional differential inclusion in a Hilbert space is studied. The results are obtained by using the fractional power of operators, multi-valued analysis, and Sadovskii's fixed point theorem. Main result gives an appropriate set of sufficient conditions for the considered system to be approximately controllable. As an example, a fractional partial nonlocal control functional differential inclusion is considered.

1. Introduction

We are concerned with the fractional delay quasilinear control inclusion

$$D_t^\alpha[u(t) - g(t, u(\sigma(t)))] \in Au(t) + \int_0^t f(t, s, B_1\mu_1(\delta(s)))ds \tag{1}$$

subject to the nonlocal control condition

$$u(0) + h(u(t)) = B_2\mu_2(t) + u_0, \tag{2}$$

where the unknown $u(\cdot)$ takes its values in a Hilbert space H with norm $\|\cdot\|$, D_t^α is the Caputo fractional derivative with $0 < \alpha \leq 1$ and $t \in J = [0, a]$. Let A be a closed linear operator defined on a dense domain $D(A)$ in H into H that generates an analytic semigroup $Q(t), t \geqslant 0$, of bounded linear operators on H and $u_0 \in D(A)$. We assume that $\{B_i : U \to H, i = 1, 2\}$ is a family of bounded linear operators, the control functions $\mu_i, i = 1, 2$, belong to the space $L^2(J, U)$, a Hilbert space of admissible control functions with U as Hilbert space, and $\sigma, \delta : J \to J'$ are delay arguments, $J' = [0, t]$. It is also assumed that $g : J \times H \to H$ and $h : C(J' : H) \to H$ are given abstract functions and $f : \Delta \times H \to H$ is a multi-valued map, $\Delta = \{(t, s) : 0 \leqslant s \leqslant t \leqslant a\}$.

Three centuries ago, fractional calculus (i.e., the calculus of non-integer order derivatives and integrals) has been dealt almost by mathematicians only. During the past decades, this subject and its potential applications have gained a lot of importance, mainly because fractional calculus has become a powerful tool with more accurate and successful results in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [5,10,13,18,22,31,33,34]. It was found that various, especially interdisciplinary applications, can be elegantly modeled with the help of fractional derivatives. Several authors have demonstrated applications in the frequency dependent damping

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behavior of viscoelastic materials [3,4], dynamics of interfaces between nanoparticles and substrates [9], the nonlinear oscillation of earthquakes [21], bioengineering [29], continuum and statistical mechanics [30], signal processing [36], filter design, robotics and circuit theory [42]. Fractional differential equations provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [39].

In control theory, one of the most important qualitative aspects of a dynamical control system is controllability. The problem of controllability consists to show the existence of a control function that steers the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. A large class of scientific and engineering problems is modeled by partial differential equations, integral equations or coupled ordinary and partial differential, integrodifferential equations, which arise in problems connected with heat-flow in materials with memory, viscoelasticity and many other physical phenomena. So it becomes important to study controllability results of such systems using available techniques. The concept of controllability plays a major role in finite-dimensional control theory, so that it is natural to try to generalize it to infinite dimensions [40]. Moreover, the exact controllability for semilinear fractional order systems, when the nonlinear term is independent of the control function, is proved by assuming that the controllability operator has an induced inverse on a quotient space, see for example [11,12]. However, if the semigroup associated with the system is compact, then the controllability operator is also compact and hence the induced inverse does not exist because the state space is infinite dimensional [50]. Thus, the concept of exact controllability is too strong and has limited applicability, while approximate controllability is a weaker concept completely adequate in applications.

In recent years, attention has been paid to establish sufficient conditions for the existence and controllability of (fractional) differential equations and inclusions, see, for instance, [1,2,7,8,28,46,51]. Ntouyas and O'Regan [35] studied existence results for semilinear neutral functional differential inclusions, Fu [20] established approximate controllability for neutral nonlocal impulsive differential inclusions and Yan [52,53] investigated the question of approximate controllability of both fractional neutral functional differential equations and fractional integro-differential inclusions with state-dependent delays. For more works about the approximate controllability for fractional systems, we refer the reader to [15,41,43–45,47,48], see also Kumar and Sukavanam [26,49]. However, in the mentioned papers, the control function is located only in the inhomogeneous part of the evolution system. For this reason, and motivated by this fact, we construct here two control functions: the first control depends on the multi-valued map and the other with the nonlocal condition. In order to realize this new complex form, we introduce here the study of approximate controllability for a class of fractional delay dynamic inclusions with nonlocal control conditions.The paper is organized as follows. In Section 2, we review some essential facts from fractional calculus, multi-valued analysis and semigroup theory, that are used to obtain our main results. In Section 3, we state and prove existence and approximate controllability results for the fractional control system (1)–(2). Finally, in Section 4, as an example, a fractional partial dynamical differential inclusion with a nonlocal control condition is considered. We end with Section 5 of conclusions and some possible future directions of research.

2. Preliminaries

In this section we give some basic definitions, notations, propositions and lemmas, which will be used throughout the work. In particular, we state main properties of fractional calculus [25,32,39], elementary principles of multi-valued analysis [16,24], and well known facts in semigroup theory [23,38,54].

Definition 1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R}^+)$ is given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the gamma function. If $a = 0$, then we can write $I^\alpha f(t) := I_0^\alpha f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{3}$$

and, as usual, $*$ denotes the convolution of functions.

Remark 2. For function (3), one has $\lim_{\alpha\to 0} g_\alpha(t) = \delta(t)$ with δ the delta Dirac function.

Definition 3. The Riemann–Liouville fractional derivative of order $n-1 < \alpha < n, n \in \mathbb{N}$, for a function $f \in C([0,\infty))$ is given by

$$^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0.$$