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# Designing multi-parameter curve subdivision schemes with high continuity

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ABSTRACT

A class of binary convergent schemes with several parameters is proposed based on eigenvalues of their difference matrices and the relation between the subdivision schemes and the difference schemes. Then multi-parameter binary subdivision schemes with arbitrary order continuity can be devised according to the conditions for $C^k$ continuity. Some existing subdivision schemes and new geometrically asymmetric ones can be obtained by setting values of parameters and a few illustrative application examples are presented. The method of designing binary multi-parameter subdivision schemes with high continuity can be generalized to design smooth $a$-ary subdivision schemes.



## 1. Introduction

Subdivision scheme defines a curve out of an initial control polygon or a surface out of an initial control mesh by subdividing them according to some refining rules recursively. Due to its efficiency and convenience, subdivision is an important subject with many applications in many fields such as Computer Graphics, Computer Aided Geometric Design, and Computer Animation. Subdivision schemes can be classified in approximating and interpolating schemes. Constructing of smooth subdivision schemes is an interesting topic in the field of subdivision scheme. In general, approximating schemes produce smoother curves or surfaces. Many methods for designing smooth subdivision schemes have been presented.

The first approximating subdivision scheme was introduced by Chaikin in [1] and its $C^1$ smoothness was proved later in [2]. The first interpolating subdivision scheme, the 4-point binary interpolatory subdivision scheme with a tension parameter which provided design flexibility, was proposed by Dyn et al. in [3] which generates $C^1$ curves. Both have been generalized to the smooth subdivision surface modeling in [4–7]. Dyn, Levin and Micchelli investigated how to increase smoothness of curves and surfaces using proper parameters in [8]. Siddiqi et al. also paid attention to the role of subdivision parameters and presented modified 3-point subdivision schemes with one parameter in [9,10] based on the results presented in [11] and other references.

Siddiqi and Ahmad also presented a 5-point approximating subdivision scheme with one parameter $w$ based on B-spline basis functions in [12], which generates $C^4$ curves when setting $w = \frac{1}{4}$. Hassan et al. presented a 4-point ternary interpolating subdivision scheme with one parameter in [13], which generates $C^2$ curves. Ghaffar and Mustafa presented a family of odd-point ternary approximating subdivision schemes with one parameter in the form of the Laurent polynomial in [14]. Zheng et al. constructed $(2n-1)$-point ternary interpolatory subdivision schemes with parameters by using variation of constants

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in [15]. Lian introduced $a$-ary subdivision schemes with a parameter for curves designing in [16], which was derived from scale functions, a notion from the content of wavelets and so on.

In [17], the authors discussed the designing of convergent subdivision schemes with two or three parameters by analyzing the eigenvalues of its difference matrix corresponding to Laurent polynomials $D(z) = (a_1z + a_0)(z + 1)^m$ and $D(z) = (a_2z^2 + a_1z + a_0)(z + 1)^m$.

In this paper we would like to explore the general case of more parameters. We firstly present some properties about the eigenvalues of the difference matrix corresponding to the Laurent polynomial

$$D(z) = (a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0)(z + 1)^m, \quad m \in N^+.$$

Then according to the relations of the subdivision schemes, the difference schemes and the conditions for $C^k$ continuity, we design multi-parameter subdivision schemes with high continuity based on eigenvalues of their difference matrices and the properties of the eigenvalues. Multi-parameter subdivision schemes have more degree of freedom than ones with one-parameter and no parameters in geometric modeling. We can control the shape of the limit curves by controlling the parameters. Moreover, if we set values of parameters suitably, we can obtain some classical subdivision schemes, for example, the Chaikin corner-cutting subdivision scheme introduced in [1], and the 4-point binary interpolatory subdivision scheme presented in [3], and some other existing subdivision schemes. We can also design some new symmetric or asymmetric smooth subdivision schemes.

## 2. Preliminaries

A general compact form of univariate binary subdivision scheme S which maps polygon $p^k = \{p_i^k\}_{i\in Z}$ to a refined polygon $p^{k+1} = \{p_i^{k+1}\}_{i\in Z}$ is defined by

$$p_j^{k+1} = \sum_{i\in Z}\alpha_{j-2i}p_i^k, \quad i \in Z,$$

where the set $\alpha = \{\alpha_j, j \in Z\}$ of coefficients is called the mask of the scheme.

The above subdivision process can be indicated as $p^{k+1} = Sp^k$ by a matrix $S$ which is called the subdivision matrix satisfying $S_{2j+i,j} = \alpha_i, i \in Z, j \in Z$. We also call the original subdivision scheme S because it is corresponding to this matrix $S$.

The polynomial $S(z) = \sum_i\alpha_iz^i$ which is determined by the mask is called the generating polynomial of the subdivision scheme S.

**Theorem 1** [18]. *Let* S *be a convergent binary subdivision scheme with mask* $\alpha$, *then*

$$\sum_j\alpha_{2j} = \sum_j\alpha_{2j+1} = 1.$$

From Theorem 1 we know that we need $S(1) = 2$ to get a convergent binary subdivision scheme.

Let $\Delta$ be backward difference operator. The set of difference vectors is

$$\Delta p^j = \{(\Delta p^j)_i = p_i^j - p_{i-1}^j, i \in Z\}.$$

We analyze the subdivision process of the difference vectors and obtain a difference scheme which can be indicated as $\Delta p^{j+1} = D\Delta p^j$, where $D$ is the subdivision matrix for this process, and we call it a difference matrix.

**Theorem 2.** *Let* S *be a subdivision scheme with the generating polynomial* $S(z)$ *and* $D(z)$ *be the generating polynomial of its difference scheme, where* $D(z) = \sum_i d_iz^i, d_i = D_{2j+i,j}, i \in Z, j \in Z$. *The relation between the subdivision matrix* $S$ *and the difference matrix* $D$ *can be indicated using the relation* $S(z) = (1 + z)D(z)$.

**Proof.** Define $p^j(z) = \sum_{i\in Z}p_i^jz^i$, which is called a generating function. Since $p_j^{k+1} = \sum_{i\in Z}\alpha_{j-2i}p_i^k$, then we have $\sum_{j\in Z}p_j^{k+1}z^j = \sum_{j\in Z}z^j\sum_{i\in Z}\alpha_{j-2i}p_i^k = \sum_{j\in Z}\alpha_{j-2i}z^{j-2i}\sum_{i\in Z}p_i^kz^{2i}$, that is $p^{k+1}(z) = S(z)p^k(z^2)$. Since $\Delta p^{j+1} = D\Delta p^j$, we have

$$\sum_i(\Delta p^{j+1})_iz^i = \sum_i(D\Delta p^j)_iz^i.$$

The left of the above equation is equal to $\sum_i(p_i^{j+1} - p_{i-1}^{j+1})z^i = (1 - z)p^{j+1}(z)$, and the right is equal to

$$\sum_i\sum_k d_{i-2k}(\Delta p^j)_kz^i = \sum_i d_{i-2k}z^{i-2k}\sum_k(p_k^j - p_{k-1}^j)z^{2k} = \sum_i d_{i-2k}z^{i-2k}\left(\sum_k p_k^jz^{2k} - z^2\sum_k p_{k-1}^jz^{2(k-1)}\right) = D(z)(1 - z^2)p^j(z^2).$$

Since both sides are equal and $p^{j+1}(z) = S(z)p^j(z^2)$, so Theorem 2 holds. □