Applied Mathematics and Computation 243 (2014) 220–231

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

General conditional recurrences

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ARTICLE INFO

Keywords:
Linear recurrences
Characteristic polynomials
Conditional recurrences
Continuants
Fibonacci sequences
Integer partitions

ABSTRACT

A general conditional recurrence sequence $\{q_n\}$ is one in which the recurrence satisfied by q_n depends on the residue of n modulo some integer $r \geqslant 2$. The properties of such sequences are studied, and in particular it is shown that any such sequence $\{q_n\}$ satisfies a single recurrence equation not dependent on the modulus r. We also obtain generating functions and Binet-like formulas for such sequences.

1. Introduction

Let $\{a_{i,j}\}$ be real numbers for $0 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant s$, and define a sequence $\{q_n\}$ with given initial terms $q_i,\ 0 \leqslant i \leqslant s-1$, and for $n \geqslant s$

$$q_n = \begin{cases} a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + \cdots + a_{0,s}q_{n-s}, & \text{if } n \equiv 0 \pmod{r}, \\ a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + \cdots + a_{1,s}q_{n-s}, & \text{if } n \equiv 1 \pmod{r}, \\ \vdots & \vdots \\ a_{r-1,1}q_{n-1} + a_{r-1,2}q_{n-2} + \cdots + a_{r-1,s}q_{n-s}, & \text{if } n \equiv r-1 \pmod{r}. \end{cases} \quad (1)$$

We call such a sequence a *general conditional recurrence sequence* with associated coefficient matrix

$$A = \begin{bmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,s} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,1} & a_{r-1,2} & \cdots & a_{r-1,s} \end{bmatrix}.$$

Some well-known sequences, such as the Fibonacci, Pell, k-Fibonacci, Tribonacci, Lucas and Jacobsthal are special cases of such sequences. The following are examples of families of sequences which are special cases of $\{q_n\}$.

(1) If $r = 2,\ s = 2$ with $a_{0,2} = a_{1,2} = 1$ and any nonzero real numbers $a_{0,1}$ and $a_{1,1}$, we obtain the *generalized Fibonacci sequences* [4].

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http://dx.doi.org/10.1016/j.amc.2014.05.108
0096-3003/

(2) If $s = 2$ and $a_{i,2} = 1$ for $0 \leqslant i \leqslant r-1$ with any nonzero real numbers $a_{i,1}$, we obtain the sequences studied in [3], the *k-periodic Fibonacci sequences*. These sequences are also studied independently in [11,13].
(3) If $s = 2$ for any non-zero real numbers $a_{i,1},\ a_{i,2},\ 0 \leqslant i \leqslant r-1$, we obtain the sequences studied in [10].
(4) If $r = 2,\ s = 2$ for real numbers $a_{i,j}$, not all zeros, we obtain the sequences defined in [12].
(5) If $a_{i,1} = a_{i,s} = 1$ and $a_{i,j} = 0,\ 0 \leqslant i \leqslant r-1$ and $2 \leqslant j \leqslant s-1$, we obtain the *s-bonacci numbers* defined on page 21 of [7].
(6) If $a_{i,j} = 1,\ 0 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant s$, we obtain the *sth order Fibonacci numbers* defined on page 21 of [7].

In addition, other examples appear in Sloane's *On-Line Encyclopedia of Integer Sequences*. If $(r,s) = (2,3)$ and $(a_{0,1}, a_{0,2}, a_{0,3}, a_{1,1}, a_{1,2}, a_{1,3}) = (1,1,0,1,1,1)$, we obtain the sequence [A068911] which can be described as the number of n step walks (each step ± 1 starting from 0) which are never more than 2 or less than -2. In the following table we give examples where $(r,s) = (3,2)$.

$(a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2})$	sequence	name
$(1,1,0,1,1,1)$	[A097564]	
$(1,1,1,0,0,0)$	[A117567]	Riordan arrays
$(0,1,1,1,1,1)$	[A092550]	two-steps-forward and one-step backward Fibonacci-based switched sequence inspired by sand piles
$(2,0,2,1,2,0)$	[A004647]	
$(2,0,1,1,1,1)$	[A133335]	

In Section 2, we present a method which finds a linear recurrence equation satisfied by $\{q_n\}$ for any given r and s. In this way, one can find properties of these sequences by using well-known results about linear recurrence sequences. In Sections 3 and 4, we study two special cases of $\{q_n\}$, namely $\{u_n\}$ and $\{v_n\}$, by taking $r = 2$ and $s = 2$, respectively. We find corresponding linear recurrence equations, for $\{u_n\}$ and $\{v_n\}$ in terms of partitions of a positive integer and generalized continuants, respectively. Also, we find the generating function and a Binet-like formula for the sequence $\{u_n\}$.

2. General conditional recurrence sequences: successor method

In this section we derive recurrence relations for sequences defined by Eq. (1). We describe a method to find the corresponding linear recurrence for a given conditional sequence: the successor method.

Let $\{t_n\}$ denote any sequence defined by a linear recurrence relation. The *successor operator*, denoted by E, is the operator defined by $Et_n = t_{n+1}$ and $E^j t_n = t_{n+j}$. A homogeneous linear recurrence relation with constant coefficients can be conveniently expressed using the operator E. The general homogeneous linear recurrence relation

$$t_{n+\ell} + a_1 t_{n+\ell-1} + a_2 t_{n+\ell-2} + \cdots + a_\ell t_n = 0,$$

where $a_1, a_2, \ldots, a_\ell$ are constants becomes $C(E)t_n = 0$ where $C(x)$ is the characteristic polynomial of degree ℓ

$$C(x) = x^\ell + a_1 x^{\ell-1} + a_2 x^{\ell-2} + \cdots + a_\ell.$$

For a more complete description of using the operator E to study recurrence sequences see [2].

Define k as the smallest positive integer such that $s \leqslant kr$. For convenience, define $a_{i,0} = 1$ for $0 \leqslant i \leqslant r-1$ and $a_{i,j} = 0$ if $s < j \leqslant kr$. We can rewrite Eq. (1) as follows:

$$0 = \begin{cases} -a_{0,0}q_n + a_{0,1}q_{n-1} + a_{0,2}q_{n-2} + \cdots + a_{0,kr}q_{n-kr}, & \text{if } n \equiv 0 \,(\mathrm{mod}\, r), \\ -a_{1,0}q_n + a_{1,1}q_{n-1} + a_{1,2}q_{n-2} + \cdots + a_{1,kr}q_{n-kr}, & \text{if } n \equiv 1 \,(\mathrm{mod}\, r), \\ \vdots & \vdots \\ -a_{r-1,0}q_n + a_{r-1,1}q_{n-1} + a_{r-1,2}q_{n-2} + \cdots + a_{r-1,kr}q_{n-kr}, & \text{if } n \equiv r-1 \,(\mathrm{mod}\, r). \end{cases} \tag{2}$$

We note that the coefficients c_{ij} in Eq. (2) have subscripts $0 \leqslant i \leqslant r-1$ and $0 \leqslant j \leqslant kr$.

If n in the ith equation of system (2) is replaced by $(n+k)r + i$ for all $0 \leqslant i \leqslant r-1$, we obtain the following r homogeneous equations:

$$\begin{aligned} -a_{0,0}q_{(n+k)r} + a_{0,1}q_{(n+k)r-1} + a_{0,2}q_{(n+k)r-2} + \cdots + a_{0,kr}q_{(n+k)r-kr} &= 0, \\ -a_{1,0}q_{(n+k)r+1} + a_{1,1}q_{(n+k)r} + a_{1,2}q_{(n+k)r-1} + \cdots + a_{1,kr}q_{(n+k)r-kr+1} &= 0, \\ \vdots \quad & \vdots \quad \vdots \\ -a_{r-1,0}q_{(n+k)r+r-1} + a_{r-1,1}q_{(n+k)r+r-2} + a_{r-1,2}q_{(n+k)r+r-3} + \cdots + a_{r-1,kr}q_{(n+k)r-kr+r-1} &= 0. \end{aligned} \tag{3}$$