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Multiple orthogonal polynomials on the semicircle and applications ☆

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ABSTRACT

In this paper two types of multiple orthogonal polynomials on the semicircle with respect to a set of r different weight functions are defined. Such polynomials are generalizations of polynomials orthogonal on the semicircle with respect to a complex-valued inner product $[f,g]=\int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta$. The existence and uniqueness of introduced multiple orthogonal polynomials for certain classes of weight functions are proved. Some properties of multiple orthogonal polynomials on the semicircle including certain recurrence relations of order $r+1$ are presented. Finally, an application in numerical integration is given.

1. Introduction

In the theory of orthogonal polynomials it is well known that a positive definite inner product generates a unique set of real orthogonal polynomials (see e.g., [6,10]). When inner product is not Hermitian, the existence of the corresponding orthogonal polynomials is not guaranteed. Gautschi and Milovanović in [8] introduced polynomials orthogonal on the semicircle with respect to the following not Hermitian inner product:

$$[f,g]=\int_0^\pi f(e^{i\theta})g(e^{i\theta})\,d\theta.$$

They proved existence and uniqueness of the corresponding orthogonal polynomials, discussed their properties and applications involving quadrature rules of Gaussian type over the semicircle, numerical differentiation, and the computation of Cauchy principal value integrals. Generalizing that work, Gautschi, Landau, and Milovanović in [7] considered orthogonality on the semicircle with respect to the suitable complex "weight function".

A generalization of orthogonal polynomials in the sense that they satisfy $r\in\mathbb{N}$ orthogonality conditions leads to the concept of multiple orthogonal polynomials. Multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite–Padé approximation of a system of r (Markov) functions. For more details about multiple orthogonal polynomials, we refer to the book by Nikishin and Sorokin [17, Chapter 4], surveys by Aptekarev [1], de Bruin [4], and Milovanović and Stanić [16], as well as the papers by Piñeiro [18], Sorokin [19–21], Van Assche [22], Van Assche and Coussement [25], Aptekarev, Branquinho, and Van Assche [2], and Chapter 23 of Ismail's book [9].

A generalization of orthogonal polynomials on the semicircle in the sense that they satisfy $r\in\mathbb{N}$ orthogonality conditions was investigated by Milovanović and Stanić in [15] (see also [13,16]), where only one type of multiple orthogonal polynomials on the semicircle with respect to so called nearly diagonal multi–index was considered. In this paper we introduce two types of multiple orthogonal polynomials on the semicircle with respect to arbitrary multi–index and prove their main

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properties. For that purpose we repeat some basic facts about polynomials orthogonal on the semicircle in Section 2 and about multiple orthogonal (real) polynomials in Section 3. Finally, in Section 4 the concept of multiple orthogonality is transferred to the semicircle, where two types of polynomials orthogonal on the semicircle are defined and their main properties are proved. Section 5 is devoted to recurrence relations. Finally, in Section 6 we present an application of introduced multiple orthogonal systems in numerical integration.

2. Polynomials orthogonal on the semicircle

For nonnegative integer n by $\mathcal{P}_n$ we denote the set of all polynomials of degree at most n, and by $\mathcal{P}$ the set of all polynomials. Let w be a weight function, which is positive and integrable on the open interval $(-1,1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$. Let us considered the following two inner products:

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx, \tag{2.1}$$

$$[f,g] = \int_{\Gamma} f(z)g(z)(iz)^{-1}w(z)\,dz = \int_0^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta, \tag{2.2}$$

where Γ is the circular part of ∂D_+ and all integrals are assumed to exist, possibly as appropriately defined improper integrals.

The inner product (2.1) is positive definite and, therefore, generates a unique set of real orthogonal polynomials $\{p_k\}$ (p_k is monic polynomial of degree k). The inner product (2.2) is not Hermitian and the existence of the corresponding orthogonal polynomials, therefore, is not guaranteed.

Definition 2.1. A system of monic complex polynomials $\{\pi_k\}$ (π_k is of degree k) is called *orthogonal on the semicircle* if $[\pi_k, \pi_\ell] = 0$ for $k \neq \ell$ and $[\pi_k, \pi_\ell] \neq 0$ for $k = \ell,\ k,\ \ell = 0,1,2,\ldots$.

We denote by m_k and μ_k the moments associated with the inner products (2.1) and (2.2), respectively,

$$m_k = (x^k, 1), \quad \mu_k = [z^k, 1], \quad k = 0,1,2,\ldots \tag{2.3}$$

Gautschi, Landau, and Milovanović in [7] have established the existence of orthogonal polynomials $\{\pi_k\}$ assuming only that

$$\operatorname{Re}\mu_0 = \operatorname{Re}[1,1] = \operatorname{Re}\int_0^{\pi} w(e^{i\theta})\,d\theta \neq 0.$$

Let $C_\varepsilon,\ \varepsilon > 0$, denotes the boundary of D_+ with small circular parts $c_{\varepsilon,\pm1}$ of radii ε and centers at ±1 spared out. We assume that w is such that

$$\lim_{\varepsilon\downarrow 0}\int_{c_{\varepsilon,\pm1}} g(z)w(z)\,dz = 0, \quad \text{for all} \quad g \in \mathcal{P}.$$

Then the following equality

$$0 = \int_C g(z)w(z)\,dz = \int_\Gamma g(z)w(z)\,dz + \int_{-1}^{1} g(x)w(x)\,dx,$$

holds for all $g \in \mathcal{P}$ (see [7]).

The monic (real) polynomials $\{p_k(z)\}$, orthogonal with respect to the inner product (2.1), as well as the associated polynomials of the second kind,

$$q_k(z) = \int_{-1}^{1} \frac{p_k(z) - p_k(x)}{z - x} w(x)\,dx, \quad k = 0,1,2,\ldots,$$

are known to satisfy a three–term recurrence relation of the form

$$y_{k+1} = (z - a_k)y_k - b_k y_{k-1}, \quad k = 0,1,2,\ldots, \tag{2.4}$$

whit initial conditions $y_{-1} = 0,\ y_0 = 1$ for $\{p_k\}$, and $y_{-1} = -1,\ y_0 = 0$ for $\{q_k\}$. Recurrence coefficient b_0 could be chosen arbitrary, but we assume that $b_0 = m_0$.

Gautschi, Landau, and Milovanović [7, Theorem 2.1] proved the existence of a unique system of monic (complex) orthogonal polynomials $\{\pi_k\}$ with respect to the inner product (2.2) and represented π_n as a linear complex combination of p_n and p_{n-1}:

$$\pi_n(z) = p_n(z) - i\theta_{n-1}p_{n-1}(z), \quad n = 0,1,2,\ldots; \quad p_{-1}(x) = 0,\ p_0(x) = 1,$$