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Parametric spline solution of the regularized long wave equation

Bin Lin

School of Mathematics and Computation Science, Zhanjiang Normal University, Zhanjiang 524048, China

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ABSTRACT

In this paper, we present a numerical method based on parametric quintic splines for the regularized long wave (RLW) equation. The truncation error is analyzed and the method shows that by choosing suitably parameters we can obtain various accuracy schemes. Stability analysis of the method is studied and the numerical results show that the method is unconditionally stable. The efficiency of the method is examined by evaluating the error norms and conservation properties of mass, energy, and momentum. The numerical simulations can validate and demonstrate the advantages of the method.

1. Introduction

The regularized long wave (RLW) equation is an important nonlinear wave equation. It can describe a lot of important physical phenomena, so it plays a major role in the study of nonlinear dispersive waves [1]. Solitary waves are wave packets or pulses propagating in nonlinear dispersive media [2]. Solitary waves interact with other solitary waves and their shapes are not affected by a collision, except for a phase shift.

The RLW equation was first proposed by Peregrine to describe the behavior of the undulate bore [3]. Analytical solutions of this problems are usually not available, especially when the nonlinear terms are involved [4,5]. Therefore, finding its numerical solutions is of practical importance. Various numerical methods has been studied to solve the equation. These include explicit multi-step method [6], non-polynomial spline method [7], conservative weighted finite difference scheme [1], Least square method [8,9] and collocation method with quadratic B-splines [10,11], differential quadrature method [12,13], Galerkin method, [14,15], integrated radial basis functions [16], finite difference methods [17], cubic B-splines [18], least square cubic B-spline finite element method [19] and mesh-free method [20].

We consider the following regularized long wave (RLW) equation.

$$u_t + u_x + \epsilon u u_x - \sigma u_{xxt} = 0, \quad a < x < b, \quad t > 0, \tag{1.1}$$

with the boundary conditions

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t), \quad u_x(a,t) = u_x(b,t) = 0, \quad t \geqslant 0 \tag{1.2}$$

and the initial condition

$$u(x,0) = f(x), \quad a \leqslant x \leqslant b, \tag{1.3}$$

E-mail addresses: linbin167@163.com, lblinbin@gmail.com

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where ε and σ are positive constants. This equation can model a large class of physical phenomena such as the nonlinear transverse waves in shallow water, ion acoustic and magnetohydrodynamic waves in plasma, and longitudinal dispersive waves in elastic rods, pressure waves in liquid–gas, bubble mixtures, and rotating flow down a tube [21,22] and so on.

The main purpose of this paper is to give a numerical method for the RLW equation, based on uniform mesh using parametric quintic splines. This paper is organized as follows. In Section 2, construction of the method is presented. Stability analysis of the method based on the von Neumann technique is given in Section 3. Section 4 is the computation of conserved quantities and errors and order of convergence. Section 5 is devoted to numerical simulations. The last section is a brief conclusion.

2. Construction of the method

A grid in the x, t plane is set up with grid points (x_i, t_j) and uniform grid spacing h and k, where $x_i = a + ih$, $h = x_{i+1} - x_i$, $i = 0, 1, 2, \ldots, N$, and $t_j = jk$, $j = 0, 1, \ldots$.

If $S_\Delta(x, t_j, \tau) = S_\Delta(x, t_j)$ is a parametric quintic spline satisfying the following differential equation

$$S_\Delta^{(4)}(x, t_j) + \tau^2 S_\Delta^{(2)}(x, t_j) = zQ_i^j + \bar{z}Q_{i-1}^j, x \in [x_{i-1}, x_i], \tag{2.1}$$

where $z = (x - x_{i-1})/h$, $\bar{z} = 1 - z$, $S_\Delta^{(2)}(x_i, t_j) = M_i^j$, $S_\Delta^{(4)}(x_i, t_j) = F_i^j$, $Q_i^j = F_i^j + \tau^2 M_i^j$ and τ is a positive parameter. A parametric quintic spline function reduces to a ordinary quintic spline as $\tau \to 0$. which satisfies the following interpolation conditions

$$S_\Delta(x_{i-1}, t_j) = u(x_{i-1}, t_j), \quad S_\Delta(x_i, t_j) = u(x_i, t_j). \tag{2.2}$$

Solving Eq. (2.1) and determining the four constants of integration using interpolation conditions, we have

$$S_\Delta(x, t_j) = zu_i^j + \bar{z}u_{i-1}^j + \frac{h^2}{3!}\left[q_3(z)M_i^j + q_3(\bar{z})M_{i-1}^j\right] + \frac{h^4}{w^4}\left[\frac{w^2}{3!}q_3(z) - q_1(z)\right]F_i^j + \frac{h^4}{w^4}\left[\frac{w^2}{3!}q_3(\bar{z}) - q_1(\bar{z})\right]F_{i-1}^j, \tag{2.3}$$

where $u_i^j = u(x_i, t_j)$, $w = h\tau$, $q_1(z) = z - \frac{\sinh(wz)}{\sinh(w)}$, $q_3(z) = z^3 - z$.

The function $S_\Delta(x, t_j)$ on the interval $[x_i, x_{i+1}]$ can be obtained with $i+1$ replacing i in Eq. (2.3).

Applying the continuity of the first and third derivatives, we obtain the following relations.

$$M_{i+1}^j + 4M_i^j + M_{i-1}^j = \frac{6}{h^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) - 6h^2(\alpha_1 F_{i+1}^j + 2\beta_1 F_i^j + \alpha_1 F_{i-1}^j), \tag{2.4}$$

$$M_{i+1}^j - 2M_i^j + M_{i-1}^j = h^2(\alpha F_{i+1}^j + 2\beta F_i^j + \alpha F_{i-1}^j), \tag{2.5}$$

where $\alpha = \frac{1}{\omega^2}[\omega cosec(\omega) - 1]$, $\beta = \frac{1}{\omega^2}[1 - \omega cot(\omega)]$, $\alpha_1 = \frac{1}{\omega^2}\left(\frac{1}{6} - \alpha\right)$, $\beta_1 = \frac{1}{\omega^2}\left(\frac{1}{3} - \beta\right)$, $i = 2, 3, \ldots, N$, and $j = 0, 1, \ldots$,

From Eqs. (2.4) and (2.5), we obtain

$$pM_{i+2}^j + qM_{i+1}^j + sM_i^j + qM_{i-1}^j + pM_{i-2}^j = \frac{1}{h^2}\left[\alpha u_{i+2}^j + 2(\beta - \alpha)u_{i+1}^j - 2(2\beta - \alpha)u_i^j + 2(\beta - \alpha)u_{i-1}^j + \alpha u_{i-2}^j\right], \tag{2.6}$$

where $p = \alpha_1 + \frac{\alpha}{6}$, $q = \frac{2\alpha+\beta}{3} - 2(\alpha_1 - \beta_1)$, $s = \frac{\alpha+4\beta}{3} + 2(\alpha_1 - 2\beta_1)$.

Consider Eq. (2.6) at two time level j and $j+1$ and subtract them, we obtain

$$\begin{aligned} &p(M_{i+2}^{j+1} - M_{i+2}^j) + q(M_{i+1}^{j+1} - M_{i+1}^j) + s(M_i^{j+1} - M_i^j) + q(M_{i-1}^{j+1} - M_{i-1}^j) + p(M_{i-2}^{j+1} - M_{i-2}^j) \\ &= \frac{1}{h^2}\left[\alpha(u_{i+2}^{j+1} - u_{i+2}^j) + 2(\beta - \alpha)(u_{i+1}^{j+1} - u_{i+1}^j) - 2(2\beta - \alpha)(u_i^{j+1} - u_i^j) + 2(\beta - \alpha)(u_{i-1}^{j+1} - u_{i-1}^j) + \alpha(u_{i-2}^{j+1} - u_{i-2}^{j+1})\right]. \end{aligned} \tag{2.7}$$

Using operator notations $Eu(x, t) = u(x + h, t)$, $Du(x, t) = u_x(x, t)$, $Iu(x, t) = u(x, t)$, $E = e^{hD}$ and expanding them in powers of hD, we obtain

$$\begin{aligned} M_i^{j+1} - M_i^j &= \left[(u_{2x})_i^{j+1} - (u_{2x})_i^j\right] - \frac{h^2}{6}\left[\frac{1}{2} + \frac{6(\alpha_1 + \beta_1)}{\alpha + \beta}\right]\left[(u_{4x})_i^{j+1} - (u_{4x})_i^j\right] \\ &+ \frac{h^4}{180}\left[\frac{1}{2} - \frac{30(2\alpha_1 - \beta_1)}{(\alpha + \beta)} + \frac{90\alpha(\alpha_1 + \beta_1)}{(\alpha + \beta)^2} + \frac{180(\alpha_1 + \beta_1)^2}{(\alpha + \beta)^2}\right]\left[(u_{6x})_i^{j+1} - (u_{6x})_i^j\right] + O(h^6), \end{aligned} \tag{2.8}$$

where $u_{ix} = \frac{\partial^i u}{\partial x^i}$,

From Eq. (2.8), when $\alpha = \frac{6}{5}(-9\alpha_1 + \beta_1)$, $\beta = -\frac{6}{5}(\alpha_1 + 11\beta_1)$, we obtain

$$u_{2x}(x_i, t_{j+1}) - u_{2x}(x_i, t_j) = M_i^{j+1} - M_i^j + O(h^6), \tag{2.9}$$

Eq. (1.1) can be rewritten as

$$\sigma\frac{\partial}{\partial t}(u_{xx} - u/\sigma) = u_x + \epsilon u u_x, \tag{2.10}$$