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A note on upper bounds for the spectral radius of weighted graphs

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ABSTRACT

Let $G = (V, E)$ be a simple connected weighted graph on n vertices, in which the edge weights are positive definite matrices. The eigenvalues of G are the eigenvalues of its adjacency matrix. In this note, we present a correction in equality part in Theorem 2 [S. Sorgun, S. Büyükköse, The new upper bounds on the spectral radius of weighted graphs, Appl. Math. Comput. 218 (2012) 5231–5238]. In addition, some related results are also provided.

1. Introduction

We only consider undirected graphs which have no loops or multiple edges. Let $G = (V, E)$ be a connected graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. A weighted graph is a graph, each edge of which has been assigned a square matrix, called the weight of the edge. All the weight matrices will be assumed to be of the same order and will be assumed to be positive definite. In particular, if the weight matrix of each of its edges is a positive number, then G is usual weighted graph. An unweighted graph is thus a weighted graph with each of the edges bearing weight 1.

Let G be a connected weighted graph on n vertices. Denote by $w_{i,j}$ the positive definite weight matrix of order p of the edge ij, and assume that $w_{i,j} = w_{j,i}$. We write $i \sim j$ if the vertices i and j are adjacent. Let $w_i = \sum_{j \in N_i} w_{i,j}$, where N_i stands for the neighbor set of vertex i.

The adjacency matrix of a weighted graph G is denoted by $A(G)$ and is defined as $A(G) = (a_{i,j})$, where

$$a_{i,j} = \begin{cases} w_{i,j} & \text{if } i \sim j, \\ 0 & \text{otherwise}. \end{cases}$$

Notice that in the definition above, the zero denotes the p-by-p zero matrix. Thus $A(G)$ is a square matrix of order np.

For any symmetric matrix B, denote by $\rho_1(B)$ the largest eigenvalue (in modulus) of B. For a connected weighted graph G, let $\gamma_i = \rho_1(w_i)$ and $\bar{\gamma}_i = \frac{\sum_{k \in N_i} \rho_1(w_{i,k}) \rho_1(w_k)}{\rho_1(w_i)}$ for all $i \in V$. For all $i, j \in V$, $w_{i,j}$ are positive definite weight matrices, which implies $\gamma_i > 0$ and $\bar{\gamma}_i > 0$ for all $i \in V$. If G is usual weighted graph, that is, edge weights are positive numbers, then $\gamma_i = w_i$ and $\bar{\gamma}_i = \bar{w}_i = \frac{\sum_{k \in N_i} w_{i,k} w_k}{w_i}$ for all $i \in V$; If G is an unweighted graph, then $\bar{\gamma}_i = \bar{m}_i = \frac{\sum_{k \in N_i} d_k}{d_i}$ is called the average of degrees of the vertices adjacent to i. The following definitions are introduced for the sake of convenience.

Definition 1.1. A graph $G = (V, E)$ is called *bipartite* if G has no cycles of odd length; the vertex set V can be partitioned into two sets V_1 and V_2 in such a way that every edge in E connects a vertex in V_1 with a vertex in V_2. For a weighted bipartite

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graph with a bipartition $V_1,\ V_2$ of V, if every vertex i in V_1 has the same $\bar{\gamma}_i$ and every vertex j in V_2 has the same $\bar{\gamma}_j$, then G is called a *weighted pseudo-semiregular bipartite graph*. If every vertex i in V of weighted graph G has the same $\bar{\gamma}_i$, then G is called a *weighted pseudo-regular graph*.

If G is an unweighted graph, then weighted pseudo-semiregular bipartite graph and weighted pseudo-regular graph are ordinary unweighted pseudo-semiregular bipartite graph and unweighted pseudo-regular graph, respectively (see [7,13]).

Upper and lower bounds for the spectral radius of unweighted graphs have been extensively investigated for a long time, reader may refer to [1,2,5–7,9,12,13] and the references therein. Recently, Upper and lower bounds for the spectral radius of weighted graphs have been studied in [3,4,10,11]. In 2012, Sorgun and Büyükköse [10] obtained

$$|\rho_1| \leqslant \max_{i\sim j}\left\{\sqrt{\bar{\gamma}_i\bar{\gamma}_j}\right\}. \tag{1}$$

Moreover, they claimed that the equality in (1) holds if and only if

(i) G is either a weighted regular graph or a weighted semiregular bipartite graph;
(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all $i,j \in V$.

However, it is not true! For example,

Example 1.2. Let $G=(V,E)$ be a weighted graph with vertex set $V=\{1,2,\ldots,9\}$ and edge set $E=\{12,14,23,25,36,45,47,56,58,69,78,89\}$. Let us take the weights of the edges as follows: $w_{12}=w_{14}=w_{23}=w_{36}=w_{47}=w_{69}=w_{78}=w_{89}=\begin{pmatrix}2&1\\1&2\end{pmatrix}$ and $w_{25}=w_{45}=w_{56}=w_{58}=\begin{pmatrix}5&4\\4&5\end{pmatrix}$. Applying the inequality (1), one has $|\rho_1|\leqslant 6\sqrt{10}$. By a direct calculation, $|\rho_1|=6\sqrt{10}$, which shows the equality in (1) holds for the weighted graph G. However, G is neither a weighted regular graph nor a weighted semiregular bipartite graph as $\rho_1(w_1)=6,\ \rho_1(w_2)=15,\ \rho_1(w_5)=36$. On the other hand, by a direct calculation, one has

$$\bar{\gamma}_i=\begin{cases}15, & i=1,3,5,7,9;\\ 24, & i=2,4,6,8.\end{cases}$$

Hence, G is a weighted pseudo-semiregular bipartite graph. It is also easy to verify that $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all $i,j\in V$.

In this note, we present a correction in the equality part in (1). It is proved that the equality in (1) holds if and only if:

(i) G is either a weighted pseudo-regular graph or a weighted pseudo-semiregular bipartite graph;
(ii) $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all $i,j\in V$.

Finally, some related results are also provided, which show that this result generalizes some known results for unweighted graphs.

2. Main results

The following Lemmas 2.1 and 2.2 are directly consequences of the Cauchy–Schwarz inequality and Lemma 2.3 in [4], respectively.

Lemma 2.1 ([4,8]). *Let B be a Hermitian n-by-n matrix with ρ_1 as its largest eigenvalue, in modulus. Then for any $\bar{x}\in\mathbb{R}^n$ $(\bar{x}\neq\bar{0})$ and $\bar{y}\in\mathbb{R}^n$ $(\bar{y}\neq\bar{0})$, the spectral radius $|\rho_1|$ satisfies*

$$|\bar{x}^TB\bar{y}|\leqslant|\rho_1|\sqrt{\bar{x}^T\bar{x}}\sqrt{\bar{y}^T\bar{y}}. \tag{2}$$

Equality holds if and only if $\bar{x}$ is an eigenvector of B corresponding to the largest eigenvalue ρ_1 and $\bar{y}=\alpha\bar{x}$ for some $\alpha\in\mathbb{R}$.

Lemma 2.2 ([4]). *Let G be a weighted graph, and let $w_{i,j}$ be the positive definite weight matrix of order p of the edge ij. Then $\gamma_i=\rho_1(w_i)>0$, where $w_i=\sum_{j\in N_i}w_{i,j}$. Moreover, let $w_{i,j}$ have a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_{i,j})$ for all $i,j\in V$. Then*

$$\gamma_i=\rho_1(w_i)=\sum_{j\in N_i}\rho_1(w_{i,j}). \tag{3}$$

Theorem 2.3. *Let G be a simple connected weighted graph and let $w_{i,j}$ be the positive definite weight matrix of order p of the edge ij. Also let ρ_1 be the largest eigenvalue, in modulus, so that $|\rho_1|$ is the spectral radius of G. Then the inequality (1) holds. Moreover, the equality in (1) holds if and only if*