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On the convergence analysis of two-step modulus-based matrix splitting iteration method for linear complementarity problems ☆

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ABSTRACT

In this paper, we analyze the convergence of the two-step modulus-based matrix splitting iteration method for the large sparse linear complementarity problems, which is proposed by Zhang (2011) [7]. The convergence conditions are presented when the system matrix is a positive definite matrix and an H_+-matrix, respectively. In particular, we establish new convergence conditions when the system matrix is an H_+-matrix.

1. Introduction

Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ be the n-dimensional real vector space and the n-by-n real matrix space, respectively. In this paper, we consider the linear complementarity problem, abbreviated as LCP (q, A), for finding a pair of real vectors w and $z \in \mathbb{R}^n$ such that

$$w := Az + q \geqslant 0, \quad z \geqslant 0 \quad \text{and} \quad z^\top w = 0, \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a given large, sparse and real matrix, and $q = (q_1, q_2, \ldots, q_n)^\top \in \mathbb{R}^n$ is a given real vector. Here, the notation '$\geqslant$' denotes the componentwise defined partial ordering between two vectors and the superscript '$\top$' denotes the transpose of a vector. For detailed descriptions about this problem and its practical backgrounds, we can see [1].

To solve the LCP (q, A) more flexible and practical in actual computation, Bai [2] proposed a class of modulus-based matrix splitting iteration methods, which includes the modulus iteration method [3], the modified modulus method [4] and the extrapolated modulus method [5,6]. Moreover, this method was extended to many methods by making use of the matrix splitting or multisplitting techniques. For example, the two-step modulus-based matrix splitting iteration method [7], the modulus-based synchronous multisplitting iteration method [8], the modulus-based synchronous two-stage multisplitting iteration method [9] and the accelerated modulus-based matrix splitting iteration method [10]. Numerical experiments have shown that these modulus-based iteration methods are powerful tools for solving the LCP (q, A).

In this paper, we study the convergence of the two-step modulus-based matrix splitting iteration method for the LCP (q, A) when the system matrix A is a positive definite matrix and an H_+-matrix, respectively. Note that Zhang in [7] only considered the system matrix A is an H_+-matrix. In particular, we give and prove the convergence theorem from different view

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when the system matrix A is an H_+-matrix. And we can see that there are some differences between our convergence conditions and the conditions of Theorem 4.2 in [7].

2. Preliminaries

In this section, we recall several necessary notations, definitions and lemmas; see [1,11–13].

For two given real m-by-n matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \geqslant B$ ($A > B$) if $a_{ij} \geqslant b_{ij}$ ($a_{ij} > b_{ij}$) holds for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is said to be nonnegative (positive) if its entries satisfy $a_{ij} \geqslant 0$ ($a_{ij} > 0$) for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Let $|A| = (|a_{ij}|) \in \mathbb{R}^{m\times n}$ be the absolute value and $A^\top$ be the transpose of the matrix A. These notations can easily be specified to vectors in $\mathbb{R}^n$.

A square matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if its off-diagonal entries are nonpositive. A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is called an M-matrix if it is a Z-matrix and $A^{-1} \geqslant 0$; and an H-matrix if its comparison matrix $\langle A\rangle = (\langle a\rangle_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix, where

$$\langle a\rangle_{ij} = \begin{cases} |a_{ij}| & \text{for } i = j, \\ -|a_{ij}| & \text{for } i \neq j, \end{cases} i,j = 1,2,\ldots,n.$$

In particular, an H-matrix having positive diagonal entries is called an H_+-matrix. Moreover, A matrix A is said to be symmetric positive definite if it is symmetric and satisfies $x^\top Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, and is said to be positive definite if its symmetric part $(A^\top + A)/2$ is positive definite; see [2,14].

Let $A \in \mathbb{R}^{n\times n}$ be a given matrix and $M, N \in \mathbb{R}^{n\times n}$ satisfy $A = M - N$. Then $A = M - N$ is called a splitting of the matrix A if M is nonsingular. The splitting $A = M - N$ is called a convergent splitting if $\rho(M^{-1}N) < 1$; an M-splitting if M is an M-matrix and $N \geqslant 0$; an H-splitting if $\langle M\rangle - |N|$ is an M-matrix; and an H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$; see [14,13,15].

$A \in \mathbb{R}^{n\times n}$ is called a P-matrix if all of its principle minors are positive. It follows that a matrix A is a P-matrix if and only if the LCP (q, A) has a unique solution for all $q \in \mathbb{R}^n$, and a nondegenerate matrix if and only if the LCP (q, A) has a finite number (possibly zero) of solutions for all $q \in \mathbb{R}^n$. A sufficient condition for the matrix A to be a P-matrix is that A is a positive definite matrix or an H_+-matrix; see [2,14,16,17].

Now, we recall basic and useful properties of a Z-matrix, an M-matrix and an H-matrix.

Lemma 2.1 [15]. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $B \in \mathbb{R}^{n\times n}$ be a Z-matrix. If $A \leqslant B$, then B is an M-matrix.*

Lemma 2.2 [18]. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and $A = D - B$, where D is the diagonal part of the matrix A. Then the following statements hold true:*

(i) *A is nonsingular and $|A|^{-1} \leqslant \langle A\rangle^{-1}$;*
(ii) *$|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.*

For a nonnegative matrix $A \in \mathbb{R}^{n\times n}$, if there exist a positive vector $v \in \mathbb{R}^n$ and two nonnegative constants $\alpha, \beta \in \mathbb{R}$ such that $\alpha v \leqslant Av \leqslant \beta v$, then $\alpha \leqslant \rho(A) \leqslant \beta$. In particular, if $\alpha v < Av < \beta v$, then $\alpha < \rho(A) < \beta$; see [13]. Thus, we can easily obtain the following lemma.

Lemma 2.3. *For a nonnegative matrix $A \in \mathbb{R}^{n\times n}$, if there exists a positive vector $v \in \mathbb{R}^n$ such that $Av < v$, then $\rho(A) < 1$.*

3. Convergence theorems

In this section, we establish the convergence theorems for the two-step modulus-based matrix splitting iteration method when the system matrix A is a positive definite matrix and an H_+-matrix.

Let $A = M_i - N_i$ ($i = 1, 2$) be two splittings of the matrix $A \in \mathbb{R}^{n\times n}$, Ω_1, Ω_2 be $n \times n$ nonnegative diagonal matrices, and Ω, Γ be $n \times n$ positive diagonal matrices such that $\Omega = \Omega_1 + \Omega_2$. If (z^*, w^*) is a solution of the LCP (q, A), then $x^* = \frac{1}{2}(\Gamma^{-1}z^* - \Omega^{-1}w^*)$ satisfies the implicit fixed-point equations

$$\begin{cases} (M_1\Gamma + \Omega_1)x^* = (N_1\Gamma - \Omega_2)x^* + (\Omega - A\Gamma)|x^*| - q, \\ (M_2\Gamma + \Omega_1)x^* = (N_2\Gamma - \Omega_2)x^* + (\Omega - A\Gamma)|x^*| - q. \end{cases} \tag{3.1}$$

Based on those and set $\Omega_1 = \Omega$, $\Omega_2 = 0$ and $\Gamma = \frac{1}{\gamma}I$, Zhang [7] presented the following two-step modulus-based matrix splitting iteration method.

Method 1 (*The two-step modulus-based matrix splitting iteration method for the LCP (q, A)*). Let $A = M_i - N_i$ ($i = 1, 2$) be two splittings of the matrix $A \in \mathbb{R}^{n\times n}$. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, compute $x^{(k+1)} \in \mathbb{R}^n$ by solving two linear systems