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Local projection stabilized method on unsteady Navier–Stokes equations with high Reynolds number using equal order interpolation

Gang Chen [a], Minfu Feng [a,*], Hong Zhou [b]

[a] School of Mathematics, Sichuan University, Chengdu, China
[b] Exploration and Development Research Institute, PetroChina Southwest Oil & Gasfield Company, China

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ABSTRACT

In this paper we propose and analyze a stabilized method for unsteady Navier–Stokes equations with high Reynolds number, using local projection stabilized method to control spurious oscillations in the velocities due to dominant convection, or in the pressure due to the velocity–pressure coupling. Using equal-order conforming elements in space and Crank–Nicolson difference in time, we derive a fully discrete formulation. We prove stability and convergence of the approximate solution. The error estimates hold irrespective of the Reynolds number, provided the exact solution is smooth. This result is comparable with the streamline diffusion and continuous interior penalty methods.

1. Introduction

The stable and accurate mixed finite element methods (FEMs) for the Navier–Stokes equations (NSEs) may suffer from violating inf–sup stability condition and oscillating approximate solutions caused by high Reynolds number. The streamline diffusion (SD) method has been a popular method to tackle these two issues in the past two decades, due to its good stability and high accuracy. It was first proposed by Brooks and Hughes [1]. Johnson et al. [2] analyzed this method and extended it to time-dependent problems using time–space elements. Johnson et al. [3] also proposed an SD method on NSEs based on a streamfunction-vorticity formulation with divergence-free discrete velocities. Hansbo and Szepessy developed a velocity–pressure SD method using time–space elements for the incompressible NSEs [4]. More related work of stabilized methods for Stokes and NSEs can be found in [5–12].

When using SD method to solve time-dependent problems, one has to solve the discretization problem on $d+1$-dimensional space–time domain. Che Sun and his coworkers proposed and developed the finite difference streamline diffusion (FDSD) method [6,7], by using finite difference discrete in time, which only need to solve the discretization problem on d-dimensional space domain. This method not only reduces the computational work, but also keeps the good features of SD method. The FDSD method was applied to unsteady NSEs in [8,9].

However, the SD/FDSD methods have some undesirable features: they introduce addition nonphysical coupling terms between velocity and pressure; they produce inaccuracy numerical solutions near the boundary; they have to calculate second derivative when using high order elements.

To overcome those disadvantages, alternative stabilized methods have been developed recently: the variational multiscale (VMS) methods [13–17], the orthogonal subscales methods [18], the continuous interior penalty (CIP) methods [19]

* Corresponding author.
E-mail address: fmf@wtjs.cn (M. Feng).

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and the local projection stabilize (LPS) methods [20–24]. These stabilized methods not only have good accuracy, but also avoid the undesirable features of SD/FDSD methods. Pressure projection [25,26] is also a favored method for increasing pressure stability. It was used to solve unsteady NSEs using equal-order elements successfully [27,28].

The subgrid scale eddy viscosity model is a numerical stabilization of convection dominated and underresolved flow. This approach adds an artificial viscosity only on the fine scales. Layton generalized this concept for the stationary convection diffusion problem [13], with error estimates almost comparable with the SD method. The subgrid scale eddy viscosity method was applied on unsteady NSEs [14,15] to derive error estimates dependent on a reduced Reynolds number (that means partially dependent on Reynolds number). This idea was used to derive a fully-discrete scheme using inf–sup unstable elements for unsteady NSEs [16,17].

Both orthogonal subscales and LPS methods involve some kinds of orthogonal property, which can be viewed as special cases of subgrid scale eddy viscosity method. The orthogonal subscales and LPS methods not only can be used to stabilize the velocity, but also can be used to stabilize pressure. In [18], orthogonal subscales method on Oseen problems was analysed, where the projection space is assumed to be continues. On the other hand, LPS method assumes the projection space to be discontinues, which was easier to be implemented in parallel computing. LPS method was first introduced to stabilize the Stokes problems [20]. Since then, several researches were done to develop the LPS method. LPS method on stationary Oseen equation was analysed in [21,22,24,23]. LPS method on stationary NSEs was also analyzed in [24], but the error estimates are dependent on Reynolds number.

It is natural to wonder how to use LPS method to solve unsteady NSEs and to obtain comparable numerical results with SD/FESD [4,12,8] and CIP [19] methods. This motivates our research. The key question for the (stabilized) FEMs for NSEs with high Reynolds number is: how to derive the error estimates held irrespective of the Reynolds number. The SD [4,12,8] and CIP [19] methods deal with this point by adding nonlinear and jump stabilized terms, respectively. In this paper, we propose and analyze a Crank–Nicolson scheme to solve unsteady NSEs with high Reynolds number, using LPS method to stabilize both velocities and pressure. Unlike the SD/FDSD methods [4,12,8] and semi-discrete CIP method [19], there are no nonlinear or jump stabilized terms introduced in our scheme. This makes our method much easier to be implemented. For the initial data we use Ritz-projection instead of L^2-projection to avoid the inaccuracy pressure close to initial moment. The almost absolute stability and error estimates held irrespective of the Reynolds number are proved. With suitable choice of parameters, our method's error estimates are quite comparable with the SD [4] and CIP [19] methods' error estimates. We implement two numerical experiments to confirm and illustrate our theoretical analysis.

An outline of the paper is as follows. In Section 2, we present necessary notations. In Section 3 we propose and analyze the stability of our method. In Section 4 we give error estimates for our scheme. In Section 5 we give some numerical experiments. In Section 6 we conclude the whole paper.

Throughout this paper, we use C to denote a positive constant independent of Δt, h and ν, not necessarily the same at each occurrence.

2. Basic notations

Let $\Omega \in \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with polygonal or polyhedral boundary $\Gamma = \partial\Omega$. Let $W^{m,p}(\Omega),\ W_0^{m,p}(\Omega)$ denote the m-order Sobolev space on Ω, $\|\cdot\|$ and $|\cdot|$ denote the norm and semi-norm on these spaces. When $p = 2$, $H_0^m(\Omega) = W_0^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,p}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,p}$, $|\cdot|_m = |\cdot|_{m,p}$. We denote the inner product of $H^m(\Omega)$ by $(\cdot,\cdot)_m$ and $(\cdot,\cdot) = (\cdot,\cdot)_0$. Let X denote a Banach space, the mapping $\phi(x,t) : [0,T] \to X$, and

$$\|\phi\|_{L^2(0,T;X)} = \left(\int_0^T \|\phi\|_X^2(t)dt\right)^{1/2}, \quad \|\phi\|_{L^\infty(0,T;X)} = \sup_{0\leqslant t\leqslant T} \|\phi\|_X(t). \tag{2.1}$$

Vector analogs of the Sobolev spaces along with vector-valued functions are denoted by upper and lower case bold face font, respectively, e.g., $\boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u}$.

Let $I = [0,T]$, where T is a fixed positive constant. The flow of an incompressible fluid is governed by the incompressible Navier–Stokes equations

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega\times I, \\ \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega\times I, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma\times I, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{in } \Omega, \end{cases} \tag{2.2}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^d$ denotes the velocities, $p = p(\boldsymbol{x},t) \in \mathbb{R}$ denotes the pressure and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t) \in \mathbb{R}^d$ denotes the body forces, $\nu = Re^{-1}$ denotes the viscosity coefficient, Re denotes the Reynolds number. Defining $\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega)$, $Q = L_0^2(\Omega) := L^2(\Omega)/\mathbb{R}$ and

$$B(\boldsymbol{u},p;\boldsymbol{v},q) = \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v},p) + (\nabla\cdot\boldsymbol{u},q), \tag{2.3}$$

$$b(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{v}) - \frac{1}{2}(\boldsymbol{w}\cdot\nabla\boldsymbol{v},\boldsymbol{u}). \tag{2.4}$$