Applied Mathematics and Computation 243 (2014) 607–610

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On edge irregularity strength of graphs

Ali Ahmad [a], Omar Bin Saeed Al-Mushayt [a], Martin Bača [b,*]

[a] Faculty of Computer Science & Information Systems, Jazan University, Postal Code 4319 Jazan, Saudi Arabia
[b] Department of Applied Mathematics and Informatics, Technical University, Letná 9, 042 00 Košice, Slovakia

ARTICLE INFO

Keywords:
Irregular assignment
Irregularity strength
Irregular total k-labeling
Edge irregularity strength

ABSTRACT

For a simple graph G, a vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called k-labeling. The weight of an edge xy in G, denoted by $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e. $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f there is $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$.

In this paper, we estimate the bounds of the edge irregularity strength and determine the exact value for several families of graphs.

1. Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a *labeling* we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V(G) \cup E(G)$ then we call the labeling *total labeling*. Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ the associated weight of a vertex $x \in V(G)$ is

$$w_\delta(x) = \sum \delta(xy),$$

where the sum is over all vertices y adjacent to x.

Chartrand et al. in [11] introduced edge k-labeling δ of a graph G such that $w_\delta(x) \neq w_\delta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. This parameter has attracted much attention [3,5,10,12,13,17,19].

Motivated by these papers, Bača et al. in [8] defined a *vertex irregular total k-labeling* of a graph G to be a total labeling of G, $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$, such that the *total vertex-weights*

$$wt(x) = \varphi(x) + \sum_{xy \in E(G)} \varphi(xy)$$

are different for all vertices, that is, $wt(x) \neq wt(y)$ for all different vertices $x, y \in V(G)$. The *total vertex irregularity strength* of G, $tvs(G)$, is the minimum k for which G has a vertex irregular total k-labeling. They also defined the total labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G one has $wt(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \neq wt(x'y') = \varphi(x') + \varphi(x'y') + \varphi(y')$. The *total edge irregularity strength*,

* Corresponding author.
E-mail addresses: ahmadsms@gmail.com (A. Ahmad), oalmushayt@yahoo.com (O.B.S. Al-Mushayt), martin.baca@tuke.sk (M. Bača).

http://dx.doi.org/10.1016/j.amc.2014.06.028

$tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. Some results on the total vertex irregularity strength and the total edge irregularity strength can be found in [1,2,4,6,7,9,15,16,18,21–23].

Combining both previous modifications of the irregularity strength, Marzuki et al. [20] introduced a new irregular total k-labeling of a graph G called *totally irregular total* k-labeling, which is required to be at the same time vertex irregular total and also edge irregular total. They gave an upper bound and a lower bound on the totally irregular total k-labeling, denoted by $ts(G)$.

The most complete recent survey of graph labelings is [14].

A vertex k-labeling $\phi : V(G) \to \{1, 2, \dots, k\}$ is called an *edge irregular* k*-labeling* of the graph G if for every two different edges e and f there is $w_\phi(e) \neq w_\phi(f)$, where the weight of an edge $e = xy \in E(G)$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $es(G)$.

In the paper, we estimate the bounds of the parameter $es(G)$ and determine the exact values of the edge irregularity strength for several families of graphs namely, paths, stars, double stars and Cartesian product of two paths.

2. Results

Our first result gives a lower bound of the edge irregularity strength.

Theorem 1. *Let G be a simple graph with maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geqslant \max\left\{\left\lceil\frac{|E(G)|+1}{2}\right\rceil, \Delta(G)\right\}.$$

Proof. Let $\phi : V(G) \to \{1, 2, \dots, k\}$ be an edge irregular k-labeling of G. Consider the weights of edges in G. The smallest among these values is at least 2 and the largest must be at least $|E(G)| + 1$. Since each edge weight is a sum of two labels, at least one label is at least $\lceil(|E(G)| + 1)/2\rceil$.

Let x be the vertex with maximum degree Δ. Since weights of all edges incident with the vertex x are distinct, also values of vertices adjacent to x are all distinct and the largest among them must be at least Δ. Therefore, $es(G) \geqslant \Delta(G)$. □

From the next two theorems it follows that the lower bound in Theorem 1 is tight.

Theorem 2. *Let P_n be a path on $n \geqslant 2$ vertices. Then $es(P_n) = \lceil n/2\rceil$.*

Proof. Let P_n be a path with the vertex set $V(P_n) = \{x_i : 1 \leqslant i \leqslant n\}$ and the edge set $E(P_n) = \{x_i x_{i+1} : 1 \leqslant i \leqslant n-1\}$. According to Theorem 1 we have that $es(P_n) \geqslant \lceil n/2\rceil$. To prove the equality, it suffices to prove the existence of an edge irregular $\lceil n/2\rceil$-labeling.

Let $\phi_1 : V(P_n) \to \{1, 2, \dots, \lceil n/2\rceil\}$ be the vertex labeling such that

$$\phi_1(x_i) = \left\lceil\frac{i}{2}\right\rceil, \quad \text{for } 1 \leqslant i \leqslant n.$$

Since $w_{\phi_1}(x_i x_{i+1}) = \phi_1(x_i) + \phi_1(x_{i+1}) = i + 1$, for $1 \leqslant i \leqslant n-1$, so the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_1 is an edge irregular $\lceil n/2\rceil$-labeling. This completes the proof. □

Theorem 3. *Let $K_{1,n}$ be a star on $n+1$ vertices, $n \geqslant 1$. Then*

$$es(K_{1,n}) = n.$$

Proof. Let x be the central vertex of the star $K_{1,n}$ and $x_1, x_2, \dots, x_n$ be the other vertices. From Theorem 1 it follows that $es(K_{1,n}) \geqslant n$. For the converse, we define a vertex n-labeling ϕ_2 as follows:

$$\phi_2(x) = 1 \quad \text{and} \quad \phi_2(x_i) = i, \quad \text{for } i = 1, 2, \dots, n.$$

We can see that the labeling ϕ_2 is an edge irregular n-labeling, which implies the assertion. □

The next theorem shows a relationship between the edge irregularity strength and the total edge irregularity strength.

Theorem 4. *Let G be a simple graph. Then $tes(G) \leqslant es(G)$.*

Proof. Let $\phi : V(G) \to \{1, 2, \dots, es(G)\}$ be an edge irregular $es(G)$-labeling. We extend the vertex labeling ϕ to the total labeling φ in such a way that