Applied Mathematics and Computation 243 (2014) 703–717

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A mixed finite element method for a time-fractional fourth-order partial differential equation ☆

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ARTICLE INFO

Keywords:
Time-fractional fourth-order PDE
Mixed finite element method
Caputo-fractional derivative
Finite difference scheme
Stability
A priori error estimates

ABSTRACT

In this paper, a numerical theory based on the mixed finite element method for a time-fractional fourth-order partial differential equation (PDE) is presented and analyzed. An auxiliary variable $\sigma = \Delta u$ is introduced, then the fourth-order equation can be split into the coupled system of two second-order equations. The time Caputo-fractional derivative is discretized by a finite difference method and the spatial direction is approximated by the mixed finite element method. The stabilities based on a priori analysis for two variables are discussed and some a priori error estimates in L^2-norm for the scalar unknown u and the variable $\sigma = \Delta u$, are derived, respectively. Moreover, an a priori error result in H^1-norm for the scalar unknown u also is proved. For verifying the theoretical analysis, a numerical test is made by using Matlab procedure.

1. Introduction

Fractional partial differential equations (FPDEs), whose theoretical analysis and numerical methods have been paid close attention by more and more math researchers, include many types based on the fractional derivative in different directions, such as space FPDEs, time FPDEs and space–time FPDEs. So far, we have found a large number of numerical methods for hunting for the numerical solutions of FPDEs. These methods include finite difference methods [1,4–6,2,9,10,12–14,16,22,23,30], spectral methods [8], finite element methods [3,11,17–21,24], mixed finite element method [7], finite volume element method [9], DG method [15] and so forth.

In recent years, finite element methods for helping people to obtain the numerical solutions for FPDEs have been increasingly concerned by most people. In the recent literatures, we find that the study of mixed finite element methods for FPDEs is very limited. So far, only a paper [7] has been studied and analyzed for a mixed finite element method of time-FPDE with second-order space derivative. However the theoretical analysis of the (mixed) finite element methods for solving the fractional fourth-order PDEs have not been mentioned and reported.

In this article, our goal is to give some detailed numerical analysis of a mixed finite element method for studying the following time-FPDE with fourth-order derivative term

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} - \Delta u + \Delta^2 u = f(\mathbf{x},t), (\mathbf{x},t) \in \Omega \times J, \tag{1.1}$$

☆ Foundation item: Supported by the National Natural Science Fund (11301258, 11361035), the Natural Science Fund of Inner Mongolia Autonomous Region (2012MS0108, 2012MS0106), the Scientific Research Projection of Higher Schools of Inner Mongolia (NJZZ12011, NJZY13199, and NJZY14013).

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http://dx.doi.org/10.1016/j.amc.2014.06.023

with boundary condition

$$u(\mathbf{x},t) = \Delta u(\mathbf{x},t) = 0, (\mathbf{x},t) \in \partial\Omega \times \bar{J}, \tag{1.2}$$

and initial condition

$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ \ \mathbf{x} \in \Omega, \tag{1.3}$$

where $\Omega \subset R^d (d \leqslant 2)$ and $J = (0,T]$ are a bounded convex polygonal domain with *Lipschitz* continuous boundary $\partial\Omega$ and the time interval with $0 < T < \infty$, respectively. $f(\mathbf{x},t)$ and $u_0(\mathbf{x})$ are given known functions and $\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}$ is defined by the following Caputo fractional derivative

$$\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x},\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha},\ \ 0 < \alpha < 1. \tag{1.4}$$

Here, we approximate the Caputo fractional derivative by a finite difference method, formulate the mixed weak formulation and fully discrete scheme, prove the stability of the fully discrete scheme and derive the theoretical analysis of some a priori error results in detail.

The remaining parts of the article is as follow. In Section 2, we introduce a finite difference method for approximating the Caputo time-fractional derivative, and then discuss the detailed proof of the truncation error. In Section 3, we formulate a fully discrete mixed scheme for the fractional fourth-order PDE (1.1) and derive the stable results for two important variables in detail. Moreover, we prove some a priori error estimates in L^2 and H^1-norms. In Section 4, we show some numerical results to illustrate the rationality and effectiveness of our method. In Section 5, we make a brief summary about the presented method and the future development.

Throughout this paper, we will denote $C > 0$ as a generic constant free of the space–time step parameters h and δ. At the same time, we define the natural inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ by $(\cdot,\cdot)$ with the corresponding norm $\|\cdot\|$. The other notations and definitions of Sobolev spaces can be easily followed in Ref. [29].

2. Approximation of time-fractional derivative

For the discretization for time-fractional derivative, let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a given partition of the time interval $[0,T]$ with step length $\delta = T/M$ and nodes $t_n = n\delta$, for some positive integer M. For a smooth function ϕ on $[0,T]$, define $\phi^n = \phi(t_n)$.

Lemma 2.1. *Assuming that $u \in C^2([0,T])$, then the time fractional derivative $\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha}$ at $t = t_{n+1}$ can be approximated by, for $0 < \alpha < 1$*

$$\begin{aligned}\frac{\partial^\alpha u(\mathbf{x},t_{n+1})}{\partial t^\alpha} &= \frac{\delta^{1-\alpha}}{\Gamma(2-\alpha)}\left[(n+1)^{1-\alpha} - (n)^{1-\alpha}\right]\frac{u^1-u^0}{\delta} \\ &\quad + \frac{\delta^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[(n-k+1)^{1-\alpha} - (n-k)^{1-\alpha}\right]\frac{3u^{k+1}-4u^k+u^{k-1}}{2\delta} + E_0^{n+1},\end{aligned} \tag{2.1}$$

where $E_0^{n+1} = E_1^0 + E_2^{n+1}$,

$$E_1^0 = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_1}\left[\left(\tau - \frac{t_1+t_0}{2}\right)\frac{\partial^2 u(\mathbf{x},t_{\frac{1}{2}})}{\partial t^2} + O((\tau - t_{\frac{1}{2}})^2) + O(\delta^2)\right]\frac{d\tau}{(t_{n+1}-\tau)^\alpha}, \tag{2.2}$$

and

$$E_2^{n+1} = \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_k}^{t_{k+1}}\left[(\tau - t_{k+1})\frac{\partial^2 u(\mathbf{x},t_{k+1})}{\partial t^2} + O((\tau - t_{k+1})^2) + O(\delta^2)\right]\frac{d\tau}{(t_{n+1}-\tau)^\alpha}. \tag{2.3}$$

Proof. Writing the integral (1.4) into two parts, we get

$$\frac{\partial^\alpha u(\mathbf{x},t_{n+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_1}\frac{\partial u(\mathbf{x},\tau)}{\partial\tau}\frac{d\tau}{(t_{n+1}-\tau)^\alpha} + \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_k}^{t_{k+1}}\frac{\partial u(\mathbf{x},\tau)}{\partial\tau}\frac{d\tau}{(t_{n+1}-\tau)^\alpha} \doteq I + II. \tag{2.4}$$

Then discretizing parts I and II by the similar approximated schemes to [8,7], respectively, we get the discrete formula (2.1) with truncation errors (2.2) and (2.3). □