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The threshold of a stochastic SIS epidemic model with vaccination ☆

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ABSTRACT

This paper is concerned with the long time behavior of a stochastic SIS epidemic model with vaccination. Sufficient conditions for extinction and persistence in mean are obtained. We find a threshold of the stochastic model which determines the outcome of the disease in case the white noises are small. At last, some numerical simulations are carried out to support our results.

1. Introduction

Epidemiology is the study of the spread of diseases with the objective to trace factors that are responsible for or contribute to their occurrence. Controlling infectious diseases has been an increasingly complex issue in recent years. In the epidemiology, vaccination is an important strategy for the elimination of infectious diseases [1,3,9–11,14,17]. Li and Ma established an SIS epidemic model with vaccination in [9]. The system has the following form:

$$\begin{cases} \dot{S}(t) = (1-q)A - \beta S(t)I(t) - (\mu+p)S(t) + \gamma I(t) + \varepsilon V(t), \\ \dot{I}(t) = \beta S(t)I(t) - (\mu+\gamma+\alpha)I(t), \\ \dot{V}(t) = qA + pS(t) - (\mu+\varepsilon)V(t). \end{cases} \tag{1.1}$$

Here $S(t)$ denotes the number of members who are susceptible to an infection. $I(t)$ denotes the number of members who are infective. $V(t)$ denotes the number of members who are immune to an infection as the result of vaccination. The parameters in the model have the following features: A denotes a constant input of new members into the population; q is a fraction of vaccinated for new borns; β represents the transmission coefficient between compartments S and I; μ and γ is the natural death rate of S, I, V compartments and recovery rate of I respectively; p is the proportional coefficient of vaccinated for the susceptible; ε is the rate of losing their immunity for vaccinated individuals; α is disease-caused death rate of infectious individuals. All parameter values are assumed to be nonnegative and $\mu, A > 0$.

According to the theory in Li and Ma [9], system (1.1) always has the disease-free equilibrium $P_0 = (S_0, I_0, V_0) = (A(\mu(1-q)+\varepsilon)/\mu(\mu+\varepsilon+p),\ 0,\ A(\mu q+p)/\mu(\mu+\varepsilon+p))$. If $R_0 \leqslant 1$, then P_0 is the unique equilibrium of

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(1.1) and it is globally stable in the invariant set Γ, where $\Gamma = \{(S, I, V) : S > 0,\ I \geqslant 0,\ V \geqslant 0,\ S + I + V \leqslant A/\mu\}$. If $R_0 > 1$, then P_0 is unstable and there is an endemic equilibrium $P^* = (S^*, I^*, V^*) = ((\mu+\gamma+\alpha)/\beta,\ \mu(\mu+\gamma+\alpha)(\mu+\varepsilon+p)(R_0-1)/\beta(\mu+\alpha)(\mu+\varepsilon),\ (qA + p(\mu+\gamma+\alpha)/\beta)/(\mu+\varepsilon))$ of system (1.1), which is globally asymptotically stable under a sufficient condition in Γ. The reproduction number of system (1.1) is

$$R_0 = \frac{A\beta(\mu(1-q)+\varepsilon)}{\mu(\mu+\gamma+\alpha)(\mu+\varepsilon+p)}. \tag{1.2}$$

In fact, epidemic models are inevitably subjected to environmental white noise. Therefore it is necessary to reveal how the noise affects the epidemic models. Stochastic differential equation (SDE) models play a significant role in various branches of applied sciences including disease dynamics, as they provide some additional degree of realism compared to their deterministic counterpart. Consequently, many authors have studied stochastic epidemic models, see [2,5–8,12,15,16,18–22]. Recently, Gray et al. [2] investigate the stochastic SIS epidemic model. They establish conditions for extinction and persistence according to the perturbation and the reproductive number R_0. In the case of persistence they show the existence of a stationary distribution and derive expressions for its mean and variance. Lahrouz and Omari [8] studied a stochastic SIRS epidemic model with general incidence rate in a population of varying size. Sufficient conditions for the extinction and the existence of a unique stationary distribution are obtained. Yang et al. [19] discussed a stochastic SIR epidemic model with saturated incidence. They utilize stochastic Lyapunov functions to show under some conditions, the solution has the ergodic property as $R_0 > 1$, and exponential stability as $R_0 \leqslant 1$. Zhao and Jiang [20] discussed the stochastic system

$$\begin{cases} \dot{S}(t) = [(1-q)A - \beta SI - (\mu+p)S + \gamma I + \varepsilon V]dt + \sigma_1 S dB_1(t), \\ \dot{I}(t) = [\beta SI - (\mu+\gamma+\alpha)I]dt + \sigma_2 I dB_2(t), \\ \dot{V}(t) = [qA + pS - (\mu+\varepsilon)V]dt + \sigma_3 V dB_3(t), \end{cases} \tag{1.3}$$

where $B_i(t)$ are independent Brownian motions, and σ_i $(i = 1, 2, 3)$ are their intensities. When the perturbations and the disease-related death rate are small, they proved that for $R_0 > 1$ there is a stationary distribution and it is ergodic, whereas asymptotic behavior of the solution around disease-free equilibrium P_0 prevails when $R_0 \leqslant 1$.

Here R_0 is the reproductive number of the deterministic model. However, compared to deterministic systems, it is difficult to give the threshold of stochastic systems.

In this paper, we further investigate the dynamics of system (1.3) and try to give a threshold $\tilde{R}_0$ which can easily determine the extinction and persistence of the disease only subjected to R_0 and σ_2.

This paper is organized as follows. In Section 2, we deduce the condition which will bring the disease to die out. The condition for the disease being persistent is given in Sections 3. In Section 4, we make simulations to confirm our analytical results. Finally, in order to be self-contained, we have the Appendix A.

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \text{Prob})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all Prob-null sets), and let $B_i(t)$ $(i = 1, 2, 3)$ be scalar Brownian motions defined on the probability space.

Remark 1.1. In [20] (Theorem 2.1), Zhao and Jiang show that SDE (1.3) exists a unique solution $Y(t) = (S(t), I(t), V(t))$ for any initial value $Y(0) = (S(0), I(0), V(0)) \in \mathbb{R}^3_+$, and $Y(t) \in \mathbb{R}^3_+$ for all $t \geqslant 0$, a.s.

2. Extinction

In this section, we will give the extinction results for system (1.3). For simplicity, define

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(r)dr.$$

Before this, let us prepare two useful lemmas.

Lemma 2.1. *Let $(S(t), I(t), V(t))$ be the solution of system (1.3) with any initial value $(S(0), I(0), V(0)) \in R^3_+$. Then*

$$\lim_{t\to\infty} \frac{S(t)+I(t)+V(t)}{t} = 0 \quad a.s.$$

Moreover

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{I(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{V(t)}{t} = 0 \quad a.s.$$

Proof. Let $u(t) = S(t) + I(t) + V(t)$. Define

$$W(u) = (1+u)^\theta,$$