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Infinitesimal bending influence on the volume change

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ABSTRACT

Applying tensor calculus we discuss change of the volume under infinitesimal bending of the surface. We obtain that the variation of the volume bounded by the surface *S* and the cone joining the origin to the boundary of the surface, under infinitesimal bending of *S* with the vector field of translation **s**, equals one third of the flux of the field **s** through the given surface *S*. An example is analysed and graphically presented. The paper points to the application of the obtained result in the calculation of the volume of a solid model.

1. Introduction

The theory of infinitesimal bending is a part of a more general bending theory, which presents one of the main consisting parts of the global differential geometry. Infinitesimal bending is a kind of deformation under which the variations of the first order of the coefficients of the first fundamental form equal zero. The magnitudes expressed by the coefficients of the first fundamental form and derivatives of these coefficients (for example Gaussian curvature, Chrisoffel's symbols, determinant of the first and the second fundamental form etc.) also have no variation of the first order. On the other hand, many other geometric magnitudes are changing under infinitesimal bending.

The problem of infinitesimal bending is in close connection with the thin elastic shell theory. The words "rigidity" and "flexibility" which describe deformed surfaces have a proper mechanical meaning. Many authors have pointed to the connection with the shell theory. A very interesting review article on the contemporary application of computer–algebra options at structural-mechanics is given in [13]. A structural analogy based on the bending theory of shells is used in the generation of smooth surfaces for engineering purposes in [22]. An interesting application of infinitesimal bending theory in architecture is given in [21].

As an important question in the theory of infinitesimal bending intrudes the question of preserving volume. The Bellows Conjecture states that *every flexible polyhedron in Euclidean 3-space preserves its volume during a flex.*

Sabitov in a note of the editor of the translation in [7, p. 231], has added a question of an infinitesimal analog of Bellows Conjecture about polyhedra and the surfaces of revolution. Alexandrov [3] has answered the conjecture negatively for polyherdra. He gave an example of non-preserving volume flexible polyhedron in the Euclidean 3-space. Also, he proved that the mentioned conjecture was true for the rotational surface of the type 0 or 1 with C^1-smooth meridian not containing a segment perpendicular to the axis of rotation. This result Velimirović extended to the piecewise-smooth surfaces of rotation in [17]. Also, she considered infinitesimal bending of a curve, as well as the variation of the volume of rotational surface with meridian that is infinitesimally flexing and proved that variation of the volume is not zero in general case [18].

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An example of a flexible polyhedron in the spherical 3-space which changes its volume during flex is given in [4]. Slutskiy in [15] gives an example of an infinitesimally nonrigid polyhedron in the Lobachevski 3-space and constructs an infinitesimal flex of that polyhedron such that the volume of the polyhedron is not stationary under the flex. The concept of infinitesimal bending further generalizes to the Riemannian space and its generalizations (see [10–12,19,23]) opening the possibility for new researches.

In this paper we discuss the variation of the volume bounded by the surface S and the cone joining the origin to the boundary of the surface, under infinitesimal bending of S. We obtain that the variation of that volume under infinitesimal bending of S with the vector field of translation **s**, equals one third of the flux of the field **s** through the given surface S. Using obtained formula, we discuss an example. We point to the application of mentioned variation in calculating the volume swept as the surface moves. The swept volume is the volume generated by the motion of an arbitrary object along an arbitrary path (or even a surface) possibly with arbitrary rotations [1]. In other words, the swept volume of a solid S is formed by all points in space which are covered by positions of S during a motion. The most frequently considered motions are those which depend smoothly on a parameter t to be thought of as time. The arising swept volumes have a variety of applications including NC machining verification, geometric modeling, robot workspace analysis, collision detection, motion planning, etc. [14].

2. Preliminaries

Let a regular surface $S \subset \mathscr{R}^3$ of class C^α, $\alpha \geqslant 3$, be given in the vector form with the equation

$$S : \mathbf{r} = \mathbf{r}(u^1, u^2), \quad (u^1, u^2) \in D \subset \mathscr{R}^2 \tag{2.1}$$

and continuous one-parameter family of the surfaces

$$\tilde{S} : \tilde{\mathbf{r}} = \mathbf{r}(u^1, u^2, \epsilon) = \mathbf{r}(u^1, u^2) + \epsilon \mathbf{z}(u^1, u^2), \quad \epsilon \in (-1, 1), \tag{2.2}$$

$$\tilde{\mathbf{r}} : D \times (-1, 1) \to \mathscr{R}^3,$$

where $\mathbf{z}(u^1, u^2) \in C^\alpha (\alpha \geqslant 3)$, is given vector field.

Definition 2.1 [8]. Infinitesimal bending of the surface S is the family of the surfaces $\tilde{S}$, (2.2), if

$$d\tilde{s}^2 - ds^2 = o(\epsilon), \tag{2.3}$$

i.e. if the difference of the squares of the line elements of this surface is of the order higher than the first with respect to ϵ. The field $\mathbf{z}(u^1, u^2)$ in (2.2) is **velocity** or **infinitesimal bending field** of the infinitesimal bending.

This definition is equivalent to the next theorem:

Theorem 2.1 [8]. *Necessary and sufficient condition for the surfaces $\tilde{S}$ (2.2) to be infinitesimal bending of the surface S (2.1) is*

$$d\mathbf{r} \cdot d\mathbf{z} = 0, \tag{2.4}$$

where $\mathbf{z}$ is the velocity field at the initial moment of deformation. □

Definition 2.2. Bending field is **trivial**, i.e. it is a field of the rigid motion of the surfaces, if it can be given in the form

$$\mathbf{z} = \mathbf{a} \times \mathbf{r} + \mathbf{b}, \tag{2.5}$$

where **a** and **b** are constant vectors. The corresponding infinitesimal bending is **trivial infinitesimal bending** or **infinitesimal motion** of the surface S.

Definition 2.3. A surface is called **(infinitesimally) rigid** if it does not allow bending fields other than the trivial ones. Otherwise the surface is called **non-rigid** or **(infinitesimally) flexible**.

According to [8], if **z** is the vector field satisfying Eq. (2.4) then there exists a unique vector field **y** so that

$$d\mathbf{z} = \mathbf{y} \times d\mathbf{r}, \tag{2.6}$$

i.e.

$$\mathbf{z}_1 = \mathbf{y} \times \mathbf{r}_1, \quad \mathbf{z}_2 = \mathbf{y} \times \mathbf{r}_2, \tag{2.7}$$

where the subscripts, 1 and 2, denote partial derivative with respect to u^1 and u^2, respectively.

Definition 2.4. The field **y** determined by (2.6), i.e. (2.7), is **rotational field** of the surface S under infinitesimal bending determined by the field **z**.

As a result of infinitesimal bending of surfaces all its line elements are rotated with vector **y**. The rotation field plays an important role in infinitesimal bending of surfaces investigations.