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Critical oscillation constant for Euler-type dynamic equations on time scales

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ABSTRACT

In this paper we study the second-order dynamic equation on the time scale $\mathbb{T}$ of the form

$$(r(t)y^{\Delta})^{\Delta} + \frac{\gamma q(t)}{t\sigma(t)} y^{\sigma} = 0,$$

where $r,\ q$ are positive rd-continuous periodic functions with $\inf\{r(t),\ t \in \mathbb{T}\} > 0$ and γ is an arbitrary real constant. This equation corresponds to Euler-type differential (resp. Euler-type difference) equation for continuous (resp. discrete) case. Our aim is to prove that this equation is conditionally oscillatory, i.e., there exists a constant $\Gamma > 0$ such that studied equation is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$.

1. Introduction

We are interested in the oscillation behavior of the second order linear dynamic equation

$$(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0 \qquad (\mathrm{L}^{\Delta}\mathrm{E})$$

on a time scale $\mathbb{T}$, where $r(t) \neq 0$ for all $t \in \mathbb{T}$. Note that this equation covers linear differential equation (frequently called as a Sturm–Liouville differential equation)

$$(r(t)y')' + p(t)(y) = 0 \qquad (\mathrm{LDE})$$

if $\mathbb{T} = \mathbb{R}$ and linear (Sturm–Liouville) difference equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0 \qquad (\mathrm{L}\Delta\mathrm{E})$$

if $\mathbb{T} = \mathbb{Z}$. Moreover, ($\mathrm{L}^{\Delta}\mathrm{E}$) is a special type of general half-linear dynamic equation

$$[r(t)\Phi(y^{\Delta})]^{\Delta} + p(t)\Phi(y^{\sigma}) = 0 \qquad (\mathrm{HL}^{\Delta}\mathrm{E})$$

if $\Phi(y) = y$. Note that generally in ($\mathrm{HL}^{\Delta}\mathrm{E}$), $\Phi(y) = |y|^{\alpha-1}\mathrm{sgn}\,y,\ \alpha > 1$. Eq. ($\mathrm{HL}^{\Delta}\mathrm{E}$) covers all of the mentioned equations.

Oscillation and non-oscillation criteria have been established at first for Eqs. (LDE) and (LΔE), see, for example [1,12,29,30], and later naturally extended on ($\mathrm{L}^{\Delta}\mathrm{E}$), ($\mathrm{HL}^{\Delta}\mathrm{E}$) and its special half-linear continuous and discrete cases, see, for example [2,4,7,8,11,22–25]. Some non-oscillatory results for Eq. ($\mathrm{L}^{\Delta}\mathrm{E}$), where the asymptotic behavior of its solutions is discussed, we can find in [3]. The asymptotic behavior results for Eq. ($\mathrm{HL}^{\Delta}\mathrm{E}$) with $r(t) \equiv 1,\ p(t) < 0$ can be found e.g. in [26,33].

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In this paper, we consider ($L^\Delta E$) in the form

$$(r(t)y^\Delta)^\Delta + \frac{\gamma q(t)}{t\sigma(t)} y^\sigma = 0 \tag{$E^\Delta E$}$$

where r, q are positive rd-continuous periodic functions, $\inf\{r(t), t \in \mathbb{T}\} > 0$ and $\gamma \in \mathbb{R}$ is an arbitrary constant. We show that ($E^\Delta E$) is the so-called *conditionally oscillatory*, that is, we prove an existence of positive constant Γ (the oscillation constant) such that ($E^\Delta E$) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. Eq. ($E^\Delta E$) is so-called Euler-type dynamic equation, because its special case (if $\mathbb{T} = \mathbb{R}$ and $r(t) \equiv p(t) \equiv 1$) is the well-known Euler differential equation

$$y''(t) + \frac{\gamma}{t^2} y = 0.$$

It was proved by Kneser in 1893, see [19], that this equations is conditionally oscillatory with the oscillation constant $\Gamma = 1/4$. The corresponding Euler difference equation

$$\Delta^2 y_k(t) + \frac{\gamma}{k(k+1)} y_{k+1} = 0$$

is also conditionally oscillatory with the same oscillation constant $\Gamma = 1/4$, see [20]. In [10,28], the Kneser result are generalized for the equation

$$(r(t)y')' + \frac{\gamma q(t)}{t^2} y = 0, \tag{EDE}$$

where r, q are positive periodic continuous functions. We can see that (EDE) is the special continuous case of ($E^\Delta E$). Further results that generalizes those in [10,28], we can find in [13,16], where the half-linear differential version of (EDE) is studied. Finally, the discrete case

$$\Delta(r_k \Delta y_k) + \frac{\gamma q_k}{k(k+1)} y_k = 0 \tag{EΔE}$$

of the Eq. ($E^\Delta E$) is studied in [14] and corresponding results for the half-linear difference version of (EΔE) we can find in [15].

Our aim is to generalize the results from [10,14], i.e., to find the oscillation constant for ($E^\Delta E$). The paper is organized as follows. In Section 2, we remind a notation on time scales and recall the basic oscillation theory for dynamic Eq. ($L^\Delta E$) with $r > 0$. Moreover, we prove some auxiliary lemmas and derive the adapted Riccati equation, which we will use later. Section 3 is devoted to announced oscillation result. Finally, in Section 4, we add some concluding remarks, examples and consequences of obtained theory.

2. Preliminaries

At the beginning, let us remind a notation on time scales. The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988, see [18], in order to unify the continuous and discrete calculus. Nowadays it is well-known calculus and often studied in applications. Remind that a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of reals. Note that $[a,b]_\mathbb{T} := [a,b] \cap \mathbb{T}$ (resp. $(a,b)_\mathbb{T} := (a,b) \cap \mathbb{T}$, $(a,b]_\mathbb{T} := (a,b] \cap \mathbb{T}$ or $[a,b)_\mathbb{T} := [a,b) \cap \mathbb{T}$) stands for an arbitrary finite time scale interval. Moreover, $[a,\infty)_\mathbb{T} := [a,\infty) \cap \mathbb{T}$, resp. $(a,\infty)_\mathbb{T} := (a,\infty) \cap \mathbb{T}$, denotes an infinite time scale interval. Symbols σ, μ, f^σ, f^Δ and $\int_a^b f(s)\Delta s$ stand for the forward jump operator, graininess, $f \circ \sigma$, Δ-derivative of f and Δ-integral of f from a to b. Further, we use the symbols $C_{rd}(\mathbb{T})$ and $C^1_{rd}(\mathbb{T})$ for the class of rd-continuous and rd-continuous Δ-differentiable functions defined on the time scale $\mathbb{T}$. See [17], which is the initiating paper of the time scale theory, and [2] containing a lot of information on time scale calculus.

Now we recall a basic elements of the oscillation theory of dynamic equations on time scales. Throughout this paper, we assume that the time scale $\mathbb{T}$ is unbounded from above, i.e., $\sup \mathbb{T} = \infty$. Consider the second order dynamic equation

$$(r(t)y^\Delta)^\Delta + p(t)y^\sigma = 0 \tag{1}$$

on a time scale $\mathbb{T}$, where $p, r \in C_{rd}(\mathbb{T})$ and $\inf\{r(t), t \in \mathbb{T}\} > 0$. Notice that any solution y of (1) satisfies $ry^\Delta \in C^1_{rd}(\mathbb{T})$.

Further note that it is not sufficient to assume only $r(t) > 0$ (instead of $\inf\{r(t), t \in \mathbb{T}\} > 0$), because it may happen that $\lim_{t \to t_0 -} r(t) = 0$ and $r(t_0) > 0$, which would not be convenient in our case. Indeed, we need $1/r \in C_{rd}(\mathbb{T})$ due to the integration of this function, which is now fulfilled, see also [21], where the similar problem is discussed.

Let us consider the initial value problem (IVP)

$$(r(t)y^\Delta)^\Delta + p(t)y^\sigma = 0, \quad y(t_0) = A, \quad y^\Delta(t_0) = B \tag{2}$$

on $\mathbb{T}$, where $A, B \in \mathbb{R}, t_0 \in \mathbb{T}$.

Theorem 1 (Existence and Uniqueness [23, p. 380]). *Let $p, r \in C_{rd}(\mathbb{T})$ and $\inf\{r(t), t \in \mathbb{T}\} > 0$. Then the IVP (2) has exactly one solution on $\mathbb{T}$.*