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Focal points and recessive solutions of discrete symplectic systems ☆

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ABSTRACT

We prove a Sturmian separation theorem comparing the number of focal points of any conjoined basis of a nonoscillatory and controllable (near ∞) symplectic difference system with the number of focal points of the recessive solution at ∞. We also present various extensions of this statement.

1. Introduction

We consider the symplectic difference system

$$z_{k+1} = \mathcal{S}_k z_k, \quad k \in \mathbb{Z}, \tag{1}$$

where $z \in \mathbb{R}^{2n}$ and the matrices $\mathcal{S} \in \mathbb{R}^{2n\times 2n}$ are *symplectic*, i.e.

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If we write $\mathcal{S}$ in the block form $\mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$, system (1) can be written in the form

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k. \tag{2}$$

Our aim is to establish a discrete version of the "singular Sturmian theorem" for nonoscillatory and controllable systems (1) which compares the number of focal points of the recessive solution with the number of focal points of any other conjoined basis. The word "singular" reflects the fact that the recessive solution appears in this statement and shows that this solution behaves, in a certain sense, as a limit for $M \to \infty$ of the so-called principal solution at $k = M$. We are motivated by the recent papers [1,12] and we extend, among others, the results of the paper [7] where a kind of singular Sturmian theory is established for linear Hamiltonian difference systems (which are a particular case of symplectic system (1)).

2. Preliminaries

The investigation of oscillatory properties of symplectic difference systems was initiated in [5], where the basic oscillation and transformation theory of (1) is established. As a starting point of our treatment, let us recall the "regular" Sturmian separation theorems for solutions of (1) as established in [8] and independently in [15]. Together with (1) we will consider its matrix version (referred to again as (1))

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$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \quad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k,$$

where X, U are real $n \times n$ matrices. A $2n \times n$ matrix solution $Z = \begin{pmatrix} X \\ U \end{pmatrix}$ of (1) is said to be a *conjoined basis* if

$$\operatorname{rank}\begin{pmatrix} X_k \\ U_k \end{pmatrix} = n \quad \text{and} \quad X_k^T U_k = U_k^T X_k. \tag{3}$$

It can be shown that if (3) holds at one index, then it holds for all indices.

The following matrices were introduced in [18]

$$\begin{cases} M_k = (I - X_{k+1} X_{k+1}^\dagger)\mathcal{B}_k, \\ T_k = I - M_k^\dagger M_k, \\ P_k = T_k^T X_k X_{k+1}^\dagger \mathcal{B}_k T_k, \end{cases}$$

here † denotes the Moore–Penrose generalized inverse of the matrix indicated. Then obviously $M_k T_k = 0$ and it can be shown (see [18]) that the matrix P_k is symmetric. The *multiplicity* of a *forward focal point* of the conjoined basis $Z = \begin{pmatrix} X \\ U \end{pmatrix}$ in the interval $(k, k+1]$ is defined as the number

$$m(k) := \operatorname{rank} M_k + \operatorname{ind} P_k,$$

where ind denotes the index, i.e., the number of negative eigenvalues, of the matrix indicated.

Similarly, following [10], we introduce the matrices

$$\begin{cases} \hat{M}_k = (I - X_k X_k^\dagger)\mathcal{B}_k^T, \\ \hat{T}_k = I - \hat{M}_k^\dagger \hat{M}_k, \\ \hat{\mathcal{P}}_k = \hat{T}_k^T X_{k+1} X_k^\dagger \mathcal{B}_k^T \hat{T}_k. \end{cases} \tag{4}$$

Using the matrices in (4), the *multiplicity* of a *backward focal* point in the interval $[k, k+1)$ is defined as the number

$$m^*(k) := \operatorname{rank} \hat{M}_k + \operatorname{ind} \hat{P}_k.$$

The definition of a backward focal point is essentially the idea of the definition of the forward focal point applied to the *reversed symplectic system*

$$z_k = \mathcal{S}_k^{-1} z_{k+1}, \quad \mathcal{S}_k^{-1} = \begin{pmatrix} \mathcal{D}_k^T & -\mathcal{B}_k^T \\ -\mathcal{C}_k^T & \mathcal{A}_k^T \end{pmatrix}. \tag{5}$$

The "regular" Sturmian separation theorems which we are going to extend in our paper read as follows. The first statement is a "reversed" version of [8, Theorem 1.3] ("reversed" means that it is Theorem 1.3 of [8] applied to the reversed system (5)), and it is proved using a variational argument (quadratic functional associated with (1) and Picone's identity). The second statement is formulated in [15] and it is proved via the comparative index of conjoined bases of (1). In both statements and later on in our paper, *all focal points are counted with their multiplicities.*

Proposition 1. *Let* $Z^{[M]} = \begin{pmatrix} X^{[M]} \\ U^{[M]} \end{pmatrix}$ *be the principal solution of* (1) *at* $k = M$*, i.e., the conjoined basis given by the initial condition* $X_M^{[M]} = 0,\ U_M^{[M]} = I$*, and let* $p_M^*(N)$ *denote its number of backward focal points in a discrete interval* $[N, M)$*. Then any other conjoined basis of* (1) *has in* $[N, M)$ *at least* $p_M^*(N)$ *backward focal points, i.e., if* $Z = \begin{pmatrix} X \\ U \end{pmatrix}$ *is any other conjoined basis and* $p^*(N)$ *denotes its number of backward focal points in* $[N, M)$*, we have the inequality*

$$p_M^*(N) \leqslant p^*(N).$$

Proposition 2. *Let* $Z^{[0]} = \begin{pmatrix} X^{[0]} \\ U^{[0]} \end{pmatrix}$, $Z^{[N]} = \begin{pmatrix} X^{[N]} \\ U^{[N]} \end{pmatrix}$ *be the conjoined bases of* (1) *given by the conditions* $X_0^{[0]} = 0,\ U_0^{[0]} = I,\ X_N^{[N]} = 0,\ U_N^{[N]} = I$*, i.e.,* $Z^{[0]}$ *and* $Z^{[N]}$ *are the principal solutions of* (1) *at* $k = 0$ *and* $k = N$*, respectively. Then the number of forward focal points of* $Z^{[0]}$ *in* $(0, N]$ *equals the number of backward focal points of* $Z^{[N]}$ *in* $[0, N)$.

Next we recall some concepts concerning behavior of solutions of (1) in infinite discrete intervals. System (1) is said to be *nonoscillatory* at $+\infty$ if there exists $M \in \mathbb{N}$ such that the principal solution of (1) at M has no forward focal point in the interval (M, ∞). Similarly, (1) is nonoscillatory at $-\infty$ if there exists M such the principal solution at $-M$ has no backward focal point in $(-\infty, -M)$. *Nonoscillation* of (1) means nonoscillation both at $+\infty$ and $-\infty$.

System (1) is said to be *controllable near* $+\infty$ if to every $M \in \mathbb{N}$ there exists $N > M, N \in \mathbb{N}$, such that for the principal solution $Z^{[M]} = \begin{pmatrix} X^{[M]} \\ U^{[M]} \end{pmatrix}$ at $k = M$ the matrix $X_N^{[M]}$ is invertible. Similarly, system (1) is controllable near $-\infty$ if to every $M \in \mathbb{N}$ there