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"Quintication" method to obtain approximate analytical solutions of non-linear oscillators

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ABSTRACT

In this paper we propose a new approach to replace nonlinear ordinary differential equations by approximate cubic–quintic Duffing oscillators in which its coefficients depend on the initial amplitude of oscillation. It is shown that this procedure leads to angular frequency values with relative errors that are lower than those found by previously developed approximate solutions.

1. Introduction

Here in this paper, we introduce an alternative approach that focus on replacing the original differential equation of motion by an equivalent cubic–quintic Duffing form representation in an attempt to obtain approximate solutions with high degree of accuracy. The main motivation comes from the fact that the usage of equivalent representation forms of the restoring force terms, in nonlinear differential equations of motion that arise in engineering and physics problems, can help us to obtain accurate approximate solutions [1–8]. For instance, Iwan developed an approach in [1] that considers an equivalent form representation of the original equations in the sense of minimum mean square difference. Sinha and Srinivasan introduced a weighted mean square method to transform the original second-order non-linear ordinary differential equation into a linear one by linearizing the non-linear terms [2]. Yuste and Sánchez [3] used a weighted mean square cubication method and the technique of Sinha and Srinivasan [2] to replace the non-linear odd restoring forces of conservative oscillators by a non-zero linear term with results that agree with numerical predictions well. By using Chebyshev polynomial expansion, Bravo Yuste replaced the non-linear oscillator restoring force by an equivalent cubic-like polynomial expression [4]. Then, he used the well-known exact solution of the undamped Duffing equation to obtain approximate solutions of conservative oscillators. Furthermore, he used the elliptic balance procedure to derive the corresponding approximate solutions of non-conservative oscillators. Belendez and co-workers used the cubication method and Chebyshev polynomial expansion to replace the rational restoring non-linear terms by an equivalent cubic polynomial equation [5–7]. By following this procedure, they found a great improvement on the accuracy of the derived approximate solutions when compare to its numerical integrations. In fact, they derived the approximate solution of the quintic Duffing-harmonic oscillator and found a maximum relative error between the exact and the approximate angular frequency values lower than 0.37% [8]. Recently, Elías-Zúñiga and Martínez-Romero used the so called enhanced cubication method to develop approximate solutions of several nonlinear oscillators and obtained approximate amplitude-time response curves and angular frequency values with maximum relative errors lower than those found by previously developed approximate solutions [9]. From these previous works and references

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cited therein, it is clear that the usage of an equivalent representation of the restoring force terms provides equations of motion whose solutions agree well with the numerical integration of the original equation of motion.

Therefore, the aim of this paper focuses on introducing a new approach, called the *quintication method*, that replaces the original non-linear ordinary differential equation by an equivalent cubic–quintic Duffing oscillator. Then, we will use the exact solution of this oscillator to obtain the approximate solution of the original equations of motion [10]. To address the effectiveness of the proposed approach, the approximate solutions of some conservative non-linear oscillators are derived. The numerical comparison between the derived approximate solutions when compare to the exact ones shows that this procedure leads to accurate angular frequency values with relative errors that are lower than those computed by previously developed approximate solutions.

2. Quintication method

In this section we introduce a new *quintication* approach and show how the approximate solutions of non-linear oscillators of the form

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{1}$$

can be obtained by replacing the original restoring force term $f(x)$, that is assumed to be an odd function, by an equivalent expression given as a function of linear, cubic and quintic terms that are found by expanding, for instance, $f(x)$ in terms of Chebyshev polynomials of the first kind [6], since these polynomials provide a uniform approximation that needs smaller number of expansion terms in comparison to the Taylor series to obtain good accuracy [11]. Therefore, we expand the restoring force $f(x)$ by using the following expression

$$f(x) = \sum_{n=0}^{N} b_{2n+1}(x_{10}) T_{2n+1}(x), \tag{2}$$

where

$$b_{2n+1} = \frac{2}{\pi}\int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} f(x) T_{2n+1}(x) dx. \tag{3}$$

If we use, for instance, the first three terms

$$T_1(x) = x; \quad T_3(x) = 4x^3 - 3x; \quad T_5(x) = 16x^5 - 20x^3 + 5x \tag{4}$$

of the Chebyshev polynomial expansion then, the equivalent restoring force $f(x)$ can be written as

$$f(x) \equiv b_1(q)T_1(y) + b_3(q)T_3(y) + b_5(q)T_5(y) \approx \alpha x + \beta x^3 + \gamma x^5, \tag{5}$$

where the coefficients $\alpha,\ \beta$, and γ depend on the form of $f(x)$, and the oscillator amplitude. Therefore, the nonlinear differential equation (1) can be replaced by an equivalent equation of the form

$$\frac{d^2x}{dt^2} + \alpha x + \beta x^3 + \gamma x^5 \approx 0. \tag{6}$$

This Eq. (6) is the well-known *cubic–quintic Duffing* equation whose exact solution, recently derived by the author in [10], is given as

$$x^2(t) = \frac{1}{a + b\,\mathrm{cn}^2(\omega t + \phi, k^2)}, \tag{7}$$

where $\mathrm{cn}(\omega t + \phi, k^2)$ is the cn Jacobian elliptic function that has a period in ωt equal to $4K(k^2)$, and $K(k^2)$ is the complete elliptic integral of the first kind for the modulus k, and $a,\ b,\ \omega$, and ϕ are constant system parameters. If we assume the following initial conditions

$$x(0) = x_{10}, \quad \dot{x}(0) = 0, \tag{8}$$

then, $\phi = 0$ and the constant parameters $a,\ b,\ k$, and ω can be determined from the following relations:

$$a = -\frac{4\gamma}{3\beta + 2\gamma x_{10}^2 \pm \sqrt{3}\sqrt{-16\alpha\gamma + (\beta - 2\gamma x_{10}^2)(3\beta + 2\gamma x_{10}^2)}}, \tag{9}$$

$$b = \frac{1 - a x_{10}^2}{x_{10}^2}, \tag{10}$$

$$k^2 = \frac{(2\alpha + \beta x_{10}^2 - 2a^2\alpha x_{10}^4 - a\beta x_{10}^4)}{2\alpha + 4a\alpha x_{10}^2 + 2\beta^2 x_{10}^2 + a\beta x_{10}^4}, \tag{11}$$

$$\omega^2 = -\frac{(2\alpha + 4a\alpha x_{10}^2 + 2\beta x_{10}^2 + a\beta x_{10}^4)}{2(1 + a x_{10}^2 + a^2 x_{10}^4)} \tag{12}$$