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Numerical approach to the Stokes problem with high contrasts in viscosity

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ABSTRACT

An algorithm based on Sherman–Morrison–Woodbury formula to solve numerically the Stokes problem is described. The algorithm allows to obtain the solution for viscosity contrasts up to ten orders of magnitude, moreover solution speed does not depend on viscosity contrast. Tests of accuracy of the algorithm are provided.

1. Introduction

The problem of high viscosity contrasts is one of the computationally challenging problems in geodynamic modeling as the fluid viscosity changes abruptly at the interface between two domains of different viscosity, and the use of standard numerical techniques leads sometimes to significant numerical errors at and in the vicinity of the interfaces (see, e.g., [1–3]). The problem of numerical handling of density and viscosity contrasts in geodynamic modeling were intensively studied in 80–90s of the XXth century by Christensen [4,5], Lenardic and Kaula [6], Naimark and Ismail-Zadeh [7,8], and Naimark et al. [9].

In simulation of two phase flows viscosity jumps are often supported on rather small sets [10]. In mathematical physics and PDE theory perturbations supported on a small set are well known under names point perturbations, finite rank perturbations and so on; first publications on the subject dates back to 1930-s, see e.g. [11]. The modern formulation of finite rank perturbations is based on the Krein operator extension theory, boundary triples and similar techniques, and allows to quite easily calculate resolvent for perturbations of partial-differential operator supported on curves, surfaces and fractals (see, e.g., [12–14] and references therein). By authors opinion inclusion of the Stokes operator into the above mentioned framework can be very fruitful, but this question will be discussed somewhere else. Few first steps in this direction were made for 2D Stokes equations with constant viscosity (see [15–17]).

In numerical computations Krein formula has its analog called Sherman–Morrison–Woodbury formula, which allows to compute inverse of a small rank perturbation to a matrix [18–21]. We propose to apply the formula to the operator for Stokes and continuity equations decomposed to two addenda: one can be inverted fast numerically using constant viscosity solver and the other is of rank proportional to viscosity jump support. The Woodbury formula allows to obtain explicit solution for variable viscosity problem, however the solution is cumbersome even for small rank perturbations. In the article we propose algorithm that can be used to obtain the solution numerically. The described method is advantageous in terms of computation speed, which does not depend on scale of viscosity jumps.

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2. Physical problem

The flow of geomaterials over geological timescales is calculated by solving the momentum equation neglecting inertial terms (Stokes equations) [22,23]

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\rho g_i. \tag{1}$$

The Stokes equations for the incompressible viscous fluid describe the balance between the external body forces and viscous stresses. The viscous force is formulated as the gradient of the stress tensor $\boldsymbol{\sigma}$ and the body force is written as the product of the fluid density ρ and the gravitational acceleration vector $\boldsymbol{g}$. Moreover, in the absence of melting and phase transitions, geodynamic flows are considered as incompressible. Incompressibility is enforced by coupling the aforementioned equations with the continuity equation

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{2}$$

where $\boldsymbol{v}$ is the velocity vector and x_i is a spatial coordinate. Eqs. (1) and (2) are valid over the model domain which we denote by Ω. The developed below method imposes no restriction on dimension of space $\mathbb{R}^d$ containing Ω; however for the sake of simplicity below we focus only on two dimensional case.

To close the system (2), the equations for the conservation of momentum and mass are supplemented with two boundary conditions. Decomposing the boundary of Ω into two non-overlapping regions, denoted by $\partial\Omega_N$ and $\partial\Omega_D$, the boundary conditions are written as

$$\sum_{j=1}^{d} \sigma_{ij} n_j = a_i, \quad \boldsymbol{x} \in \partial\Omega_N, \quad i = 1, \ldots, d \tag{3}$$

and

$$\boldsymbol{v} = \boldsymbol{b}, \quad \boldsymbol{x} \in \partial\Omega_D. \tag{4}$$

Here $\boldsymbol{n}$ is the outward point normal to the boundary of Ω, $\boldsymbol{a}$ is an applied traction and $\boldsymbol{b}$ is a prescribed velocity.

The mechanical behavior of the material is defined by a constitutive relationship. We relate the stress tensor $\boldsymbol{\sigma}$ to the strain rate tensor $\boldsymbol{\epsilon}$, using a linear, isotropic viscous rheology given by

$$\sigma_{ij} = -p\delta_{ij} + 2\eta\dot{\epsilon}_{ij}, \tag{5}$$

where δ is the Kronecker delta, p is the pressure, η is the viscosity and the strain rate is given by

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{6}$$

Prior to any discretization, the equations above are nondimensionalized by means of dynamic scaling. This scaling is achieved by first defining a set of characteristic units such as a characteristic length (e.g., domain size), a characteristic time (e.g., inverse background strain rate), a characteristic viscosity (e.g., minimum viscosity in the domain) and secondly deriving all the related characteristic units (mass, stress, force...). We employ characteristic units that are equal to 1. The results are not scaled back to dimensional units and therefore the velocity errors and pressure are dimensionless.

3. Discretization

We solve Eqs. (1) and (2) for the primitive variables v_i and p. The governing equations can be written as follows

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right) - \frac{\partial p}{\partial x_i} = -\rho g_i, \quad i = 1, \ldots, d. \tag{7}$$

$$\sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i} = 0. \tag{8}$$

We use finite-difference discretization on non-staggered grid:

$$\begin{cases} h_j^{-1}\nabla_j\left(\eta\left(h_j^{-1}\Delta_j v_i + h_i^{-1}\Delta_i v_j\right)\right) + \sum_{j\neq i} h_j^{-1}\Delta_j\left(\eta\left(h_j^{-1}\nabla_j v_i + h_i^{-1}\nabla_i v_j\right)\right) - h_i^{-1}\nabla_i p = -\rho g_i, \\ \sum_{i=1}^{d} h_i^{-1}\Delta_i v_i = 0, \end{cases} \tag{9}$$