Applied Mathematics and Computation 235 (2014) 105–109

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Monotonicity-preserving doubly infinite matrices

F. Aydin Akgun [a], B.E. Rhoades [b,*]

[a] Department of Mathematical Engineering, Yildiz Technical University, 34210 Esenler, Istanbul, Turkey
[b] Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA

ARTICLE INFO

Keywords:
Monotonicity
Doubly infinite matrices
Doubly infinite Hausdorff matrices

ABSTRACT

We prove that a regular nonnegative doubly infinite matrix preserves all monotonicities of double sequences if and only if it is a conservative double Haudorff matrix with all nonnegative entries.

Let $A = (a_{mnkj})$ be a doubly infinite matrix. Then A is called a double triangle if $a_{mnjk} = 0$ for all $j > m$ or $k > n$. Let $x = \{x_{mn}\}$ be a sequence of real numbers, and define the forward difference operators Δ_{10}^m and Δ_{01}^n by

$$\Delta_{10}x_{jk} = x_{jk} - x_{j+1,k}, \quad \Delta_{10}^{m+1} = \Delta_{10}(\Delta_{10}^m x_{jk})$$

and

$$\Delta_{01}x_{jk} = x_{jk} - x_{j.k+1}, \quad \Delta_{01}^{n+1}x_{jk} = \Delta_{01}(\Delta_{01}^n x_{jk}).$$

Using an induction argument it can be shown that

$$\Delta_{10}^m \Delta_{01}^n x_{jk} = \sum_{i=0}^{m}\sum_{\ell=0}^{n}(-1)^{i+\ell}\binom{m}{i}\binom{n}{\ell}x_{j+i,k+\ell}, \quad m,n = 0,1,2,\ldots. \tag{1}$$

The matrix A is said to preserve all monotonicities if any kind of monotonic behavior is transferred to $y = \{y_{mn}\}$, where

$$y_{mn} = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{mnjk}x_{jk}.$$

Also, if, for any $m,n \geqslant 0$, $\Delta_{10}^m \Delta_{01}^n x_{jk} \geqslant 0$ (in which case we shall say that x is mn-convex) then one must have $\Delta_{10}^m \Delta_{01}^n y_{jk} \geqslant 0$ for all i and j.

A double Hausdorff matrix is a double triangle with nonzero entries

$$h_{mnjk} = \binom{m}{j}\binom{n}{k}\Delta_{10}^{m-j}\Delta_{01}^{n-k}\mu_{jk}, \tag{2}$$

where $\{\mu_{jk}\}$ is an arbitrary sequence of real or complex numbers, and the differences are defined by (1).

Since we will be concerned with preserving monotonicities we shall use only real number sequences.

Adams [1] has stated that, for any double Hausdorff matrix,

* Corresponding author.
E-mail addresses: fakgun@yildiz.edu.tr (F. Aydin Akgun), rhoades@indiana.edu (B.E. Rhoades).

http://dx.doi.org/10.1016/j.amc.2014.02.072
0096-3003/

$$\sum_{j=0}^{m}\sum_{k=0}^{n} h_{mnjk} = \mu_{00}.$$

Using (1) and (2) can be written in the form

$$h_{mnjk} = \binom{m}{j}\binom{n}{k}\sum_{i=0}^{m-j}\sum_{\ell=0}^{n-k}\binom{m-j}{i}\binom{n-k}{\ell}(-1)^{i+\ell}\mu_{i+j,\ell+k}. \tag{3}$$

If one defines the double matrix δ by

$$\delta_{mnjk} = (-1)^{k+j}\binom{m}{j}\binom{n}{k},$$

Lemma 1. *A double Hausdorff matrix H has the representation*

$$H = \delta\lambda\delta,$$

where $\lambda = (\lambda_{mnjk})$ is the double diagonal matrix with $\lambda_{mnjk} = 0$ except when $j = m$ and $k = n$, and $\lambda_{mnmn} = \mu_{mn}$.

Proof.

$$(\lambda\delta)_{mnrs} = \sum_{i=r}^{m}\sum_{j=s}^{n}\lambda_{mnij}\delta_{ijrs} = \mu_{mn}\Delta_{mnrs}$$

and

$$(\delta\lambda\delta)mnjk = \sum_{i=j}^{m}\sum_{\ell=k}^{n}(-1)^{i+\ell}\binom{m}{i}\binom{n}{\ell}\mu_{i\ell}\binom{i}{j}\binom{\ell}{k}.$$

Setting $r = i - j$ and $s = \ell - k$,

$$(\delta\lambda\delta)_{mnjk} = \sum_{r=0}^{m-j}\sum_{s=0}^{n-k}(-1)^{r+j+s+k}\binom{m}{r+j}\binom{n}{s+k}\mu_{r+j,s+k}\binom{r+j}{j}\binom{s+k}{k}.$$

Note that

$$\binom{m}{r+j}\binom{r+j}{j} = \binom{m}{j}\binom{m-j}{r}$$

and

$$\binom{n}{s+k}\binom{s+k}{k} = \binom{n}{k}\binom{n-k}{s}.$$

Therefore

$$(\delta\lambda\delta)_{mnjk} = \binom{m}{j}\binom{n}{k}\sum_{r=0}^{m-j}\sum_{s=0}^{n-k}(-1)^{r+j}\binom{m-j}{r}\binom{n-k}{s}\mu_{r+j,s+k} = h_{mnjk}. \qquad \square$$

Lemma 2. $\delta^{-1} = \delta$.

Proof. It will be sufficient to prove that $\delta^2 = I$.

$$\begin{aligned}\delta^2_{mnjk} &= \sum_{i=j}^{m}\sum_{\ell=k}^{n}\delta_{mni\ell}\delta_{i\ell jk} = \sum_{i=j}^{m}\sum_{\ell=k}^{n}(-1)^{i+\ell}\binom{m}{i}\binom{n}{\ell}(-1)^{j+k}\binom{i}{j}\binom{\ell}{k} = \sum_{r=0}^{m-j}\sum_{s=0}^{n-k}(-1)^{r+s}\binom{m}{r+j}\binom{n}{s+k}\binom{j+r}{j}\binom{s+k}{k}\\ &= \binom{m}{j}\binom{n}{k}\sum_{r=0}^{m-j}\sum_{s=0}^{n-k}(-1)^{r+s}\binom{m-j}{r}\binom{n-k}{s} = \binom{m}{j}\binom{n}{k}(1-1)^{m-j}(1-1)^{n-k} = \begin{cases}1, & j=m,\ n=k\\ 0, & \text{otherwise.}\end{cases} \qquad \square\end{aligned}$$

Lemma 3. *Suppose that a, b and c are real sequences such that*

$$\Delta_{10}^{m}\Delta_{01}^{n}a_{00} = b_{mn}\Delta_{10}^{m}\Delta_{01}^{n}c_{00}. \tag{4}$$