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Numerical study of the porous medium equation with absorption, variable exponents of nonlinearity and free boundary

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ABSTRACT

In this paper, we study an application of the moving mesh method to the porous medium equation with absorption and variable exponents of nonlinearity in 2D domains with moving boundaries.

The boundary's movement is governed by an equation prompted by the Darcy law and the spatial discretization is defined by a triangulation of the domain. At each finite element, the solution is approximated by piecewise polynomial functions of degree $r \geqslant 1$ using Lagrange interpolating polynomials in area coordinates. The vertices of the triangles move according to a system of differential equations which is added to the equations of the problem. The resulting system is converted into a system of ordinary differential equations in time variable, which is solved using a suitable integrator. The integrals that arise in the system of ordinary differential equations are calculated using the Gaussian quadrature. Finally, we present some numerical results of application of this technique.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz-continuous boundary $\partial\Omega$ and let $\Omega_T = \Omega \times]0,T]$ be a cylinder of height $T < \infty$. We consider the nonnegative solutions of the following problem:

$$\frac{\partial u}{\partial t} = div(u^{\gamma(\mathbf{x},t)}\nabla u) - u^{\sigma(\mathbf{x},t)}, \quad \text{in } \Omega_T, \tag{1}$$

$$u = 0 \quad \text{on } \Gamma_T = \partial\Omega \times [0,T], \tag{2}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{in } \Omega, \tag{3}$$

where γ and σ are bounded functions defined on Ω_T such that

$$0 \leqslant \gamma(\mathbf{x},t) \leqslant \gamma^+ < \infty, \quad 1 \leqslant \sigma(\mathbf{x},t) \leqslant \sigma^+ < \infty, \quad \forall \mathbf{x} \in \bar{\Omega}_T. \tag{4}$$

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Equations of type (1) appear in continuum mechanics [1] as a model of the motion of an ideal barotropic gas through a porous medium, where the pressure is assumed to depend explicitly on the density and on the temperature. There are very few results on variable exponents; we refer to Antontsev and Shmarev [1] and to Lian et al. [2]. They proved the existence and uniqueness of weak solutions and showed that the solutions to this problem have the finite speed of propagation property. If the initial solution has compact support, then a free boundary delimiting the support of the solution arises and hence we can treat the problem as a free boundary problem, considering that $\Omega(t)$ is the region occupied by the gas at time t.

If $\gamma(\mathbf{x},t)$ and $\sigma(\mathbf{x},t)$ do not depend on $\mathbf{x}$ and t, then we have the classical porous media equation:

$$\frac{\partial u}{\partial t} = \Delta(u^m) - u^p.$$

In this case, there exists an abundant literature and we refer, for example, to the survey book of Vazquez [3]. For constant exponents in one spatial dimension, the first results are due to Kalashnikov [4], who demonstrates the existence and uniqueness of weak solutions. He also proved that in the range $p > 1$ the weak solutions have the finite speed of propagation property. Knerr [5] showed that $\Omega(t) = \{x : u(x,t) > 0\}$ is an interval $]\zeta_1(t), \zeta_2(t)[$ with $\zeta_i \in C^0([0,\infty[) \cap C^{0,1}(]0,\infty[)$ in the range $m > 1, p > 1$. Using these results, Herrero and Vazquez [6] gave conditions for the existence of a waiting time t^* and proved that $\zeta_i \in C^1(]t^*,\infty[)$. In addition, they showed that, for $p > m,\ \zeta_i(t)$ satisfies

$$\zeta_i'(t) = -\frac{m}{m-1} u_x^{m-1}(\zeta_i(t),t), \quad t \in]t^*,T]. \tag{5}$$

Eq. (5) should be understood as a limit when x tends to the boundary. Later, Shmarev and Vazquez [7] proved that Eq. (5) is also valid for $1 < p \leqslant m$.

For dimensions greater than one, Shmarev [8,9] gave the equation

$$\Gamma'(t) = \lim_{\mathbf{x}\to\Gamma}\left(-\frac{m}{m-1}\nabla u^{m-1} + \nabla\Pi\right) \tag{6}$$

for the boundary's velocity, in the range $m > 1,\ p > 0$ and $m + p \geqslant 2$. In (6), Π is a solution of the elliptic problem

$$\begin{cases} div(u\nabla\Pi) = u^p & \text{in } \Omega(t), \\ \Pi = 0 & \text{on } \Gamma(t) \end{cases}$$

with t as a parameter. Moreover, he proved that u and $\Gamma = \partial\Omega$ are real analytical functions in t and that they preserve the initial regularity in the spatial variables.

Since we intend to capture the free boundary, it is particularly useful to use a method where the mesh can adapt to this movement. The finite element method (FEM) with adaptive mesh is widely used to solve differential equations, in fixed domains, that have solutions with large variations in the domain. It is well-known that best results in accuracy and efficiency can be obtained if the method locates the grid points in regions where the solution needs further definition.

If there is a guarantee that the number of finite elements is sufficient to describe well the variations of the solution and that the movement of the nodes is suitable, we obtain an extremely efficient method. The main difficulty lies in the choice of the movement of the nodes such that there are no singularities. Some methods with adaptive mesh have been proposed; for example, the moving finite element method of Miller [10], the weighted moving finite element method studied by Carlson and Miller [11,12], the string gradient weighted moving finite element deduced by Wacher et al. [13] in 2005, the moving mesh finite element method of Baines et al. [14–16], the moving mesh for partial differential equations (MMPDE) deduced by Huang et al. [17]. The MMPDE uses the equidistribution principle [18] to derive various methods, introducing a monitor function to define the movement of the mesh.

The aim of this work is to study and to implement, in Matlab, the MMPDE for finding approximate solutions to the porous medium equation with absorbtion (PMEA) in spatial 2-dimensional domains with moving boundaries. The technique uses the finite element method with a moving mesh and approximations of degree greater than one. The use of piecewise polynomial basis functions of degree $r \geqslant 1$ allows the reduction of the number of finite elements, while maintaining the required accuracy. The boundary is calculated using a discretized version of the normal velocity. A PDE system, which involves the velocity of the nodes, is used to move the mesh to certain regions of the domain. The equations of the mesh, the boundary and the physical problem, are solved simultaneously.

The resulting system is converted into a system of ordinary differential equations in time variable, which is solved using a suitable integrator. The integrals that arise in the system of ordinary differential equations are calculated using the Gaussian quadrature.

We start by presenting the exact formulation of the problem. In Section 3, we describe how the spatial discretization of the domain is done. Next, we show the construction of the adaptive mesh method and, in Section 5, the approximate solution is obtained. The equations to the movement of the boundary are discretized in Section 6. Finally, in Section 7, we test the method by presenting some numerical results of application to physical problems.

2. Statement of the problem

In order to well define the problem, we need to obtain an equation for the velocity of the boundary, $v = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)$.