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Total edge irregularity strength of generalized prism

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ABSTRACT

The *generalized prism* P_n^m can be defined as the Cartesian product $C_n \square P_m$ of a cycle on n vertices with a path on m vertices. An edge irregular total k-labeling of a graph G is such a labeling of the vertices and edges with labels $1, 2, \ldots, k$ that the weights of any two different edges are distinct, where the weight of an edge is the sum of the label of the edge itself and the labels of its two end vertices. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength, $tes(G)$.

In this paper we determine the exact value of the total edge irregularity strength of the generalized prism P_n^m.

1. Introduction

In [3] the authors defined the notion of an edge irregular total k-labeling of a graph $G = (V, E)$ to be a labeling of the vertices and edges of G $f : V \cup E \to \{1, 2, \ldots, k\}$ such that, the *edge weights* $wt_f(xy) = f(x) + f(xy) + f(y)$ are different for all edges, i.e. $wt_f(xy) \neq wt_f(x'y')$ for all edges $xy, x'y' \in E$ with $xy \neq x'y'$. They also defined the *total edge irregularity strength of G*, $tes(G)$, as the minimum k for which the graph G has an edge irregular total k-labeling.

The total edge irregularity strength is an invariant analogous to irregular assignments and the irregularity strength of a graph G introduced by Chartrand et al. [6] and studied by numerous authors, see [4,11,12,15,22]. An *irregular assignment* is a k-labeling of the edges $\varphi : E \to \{1, 2, \ldots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G, and the smallest k for which there is an irregular assignment is the *irregularity strength*, $s(G)$.

The corresponding problem where only adjacent vertices are required to have different weights, i.e. the weights form a proper vertex coloring of the graph, was introduced by Karoński et al. in [18]. They conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$, such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different. The current record is that 16 labels suffice, see [1].

In [3] is given a lower bound on the total edge irregularity strength of a graph:

$$tes(G) \geqslant \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}, \qquad (1)$$

where $\Delta(G)$ is the maximum degree of G.

Ivančo and Jendroľ [14] posed the following conjecture:

Conjecture 1 [14]. Let G be an arbitrary graph different from K_5. Then

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$$tes(G) = \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}. \quad (2)$$

Conjecture 1 has been verified for trees [14], for complete graphs and complete bipartite graphs [16,17], for the Cartesian product of two paths $P_n \square P_m$ [21], for corona product of a path with certain graphs [23], for large dense graphs with $\frac{|E(G)|+2}{3} \leqslant \frac{\Delta(G)+1}{2}$ [5], for toroidal grid [9] and for strong product of two paths [2].

In the present paper, we investigate the total edge irregularity strength of the generalized prism. The *generalized prism* P_n^m can be defined as the Cartesian product $C_n \square P_m$ of a cycle on n vertices with a path on m vertices. If we consider a cycle C_n with $V(C_n) = \{x_i : 1 \leqslant i \leqslant n\}, E(C_n) = \{x_i x_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{x_n x_1\}$ and a path P_m with $V(P_m) = \{y_j : 1 \leqslant j \leqslant m\}, E(P_m) = \{y_j y_{j+1} : 1 \leqslant j \leqslant m-1\}$, then $V(P_n^m) = V(C_n \square P_m) = \{(x_i, y_j) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ is the vertex set of the graph P_n^m and $E(P_n^m) = E(C_n \square P_m) = \{(x_i, y_j)(x_{i+1}, y_j) : 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m\} \cup \{(x_n, y_j)(x_1, y_j) : 1 \leqslant j \leqslant m\} \cup \{(x_i, y_j)(x_i, y_{j+1}) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m-1\}$ is the edge set of P_n^m. So, $|V(P_n^m)| = nm$ and $|E(P_n^m)| = n(2m-1)$.

The generalized prism P_n^m has been studied extensively in recent years. Kuo et al. [19] and Chiang et al. [8] studied distance-two labelings of $C_n \square P_m$. Deming et al. [10] gave complete characterization of the Cartesian product of cycles and paths for their incidence chromatic numbers. Gravier et al. [13] showed the link between the existence of perfect Lee codes and minimum dominating sets of $C_n \square P_m$. Lai et al. [20] determined the edge addition number for the Cartesian product of a cycle with a path. Chang et al. [7] established upper bounds and lower bounds for global defensive alliance number of $C_n \square P_m$ and showed that the bounds are sharp for certain n, m.

In this paper we determine the exact value of $tes(C_n \square P_m)$ and we add further support to Conjecture 1.

2. Main result

Before we proceed to our main result we discuss the total edge irregularity strength for small case.

Lemma 1. *Let $m \geqslant 2$. Then $tes(C_3 \square P_m) = 2m$.*

Proof. From (1) it follows that $tes(C_3 \square P_m) \geqslant \lceil \frac{6m-1}{3} \rceil = 2m$. The existence of the optimal labeling proves the converse inequality.

For $1 \leqslant j \leqslant m$ we define

$$f_1((x_i, y_j)) = \begin{cases} 2m, & \text{if } i = 1 \\ 1, & \text{if } i = 2 \\ m, & \text{if } i = 3 \end{cases}$$

$$f_1((x_i, y_j)(x_{i+1}, y_j)) = j \quad \text{for } i = 1, 2,$$

$$f_1((x_3, y_j)(x_1, y_j)) = m + j$$

and for $1 \leqslant j \leqslant m-1$ we define

$$f_1((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} m + j, & \text{if } i = 1 \\ j, & \text{if } i = 2 \\ m + 1 + j, & \text{if } i = 3 \end{cases}$$

One can check that all vertex and edge labels are at most $2m$. Moreover under the labeling f_1 for weights of the edges we have

$$wt_{f_1}((x_2, y_j)(x_2, y_{j+1})) = 2 + j \quad \text{for } 1 \leqslant j \leqslant m-1,$$
$$wt_{f_1}((x_2, y_j)(x_3, y_j)) = m + 1 + j \quad \text{for } 1 \leqslant j \leqslant m,$$
$$wt_{f_1}((x_1, y_j)(x_2, y_j)) = 2m + 1 + j \quad \text{for } 1 \leqslant j \leqslant m,$$
$$wt_{f_1}((x_3, y_j)(x_3, y_{j+1})) = 3m + 1 + j \quad \text{for } 1 \leqslant j \leqslant m-1,$$
$$wt_{f_1}((x_3, y_j)(x_1, y_j)) = 4m + j \quad \text{for } 1 \leqslant j \leqslant m,$$
$$wt_{f_1}((x_1, y_j)(x_1, y_{j+1})) = 5m + j \quad \text{for } 1 \leqslant j \leqslant m-1.$$

We can see that weights of the edges form a sequence of consecutive integers from 3 up to $6m-1$. Thus the labeling f_1 is the desired edge irregular total $2m$-labeling. This concludes the proof. □

Let $k = \left\lceil \frac{n(2m-1)+2}{3} \right\rceil = \left\lfloor \frac{n(2m-1)+4}{3} \right\rfloor$. Then

$$\frac{n(2m-1)+2}{3} \leqslant k \leqslant \frac{n(2m-1)+4}{3}. \quad (3)$$

Let us present the following useful lemma.