Applied Mathematics and Computation 235 (2014) 187–200

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Asymptotic expansions of integral means and applications to the ratio of gamma functions

Neven Elezović *, Lenka Vukšić

Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia

ARTICLE INFO

Keywords:
Asymptotic expansion
Integral mean
Gamma function
Wallis ratio
Digamma function

ABSTRACT

Integral means are important class of bivariate means. In this paper we prove the very general algorithm for calculation of coefficients in asymptotic expansion of integral mean. It is based on explicit solving the equation of the form $B(A(x)) = C(x)$, where asymptotic expansions of B and C are known. The results are illustrated by calculation of some important integral means connected with gamma function.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and let f be a strictly monotone continuous function on I and $s, t \in I, s < t$. Then there exists the unique $\vartheta \in [s, t]$ for which

$$\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u = f(\vartheta).$$

ϑ is called *integral f-mean* of s and t, and is denoted by

$$I_f(s,t) = f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right), \tag{1.1}$$

see [1,2] for details.

Many classical means can be interpreted as integral means, for suitably chosen function f. For example,

$$f(x) = x, \quad I_x(s,t) = \frac{s+t}{2} = A(s,t),$$

$$f(x) = \log x, \quad I_{\log x}(s,t) = \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}} = I(s,t),$$

$$f(x) = \frac{1}{x}, \quad I_{1/x}(s,t) = \frac{s-t}{\log s - \log t} = L(s,t),$$

$$f(x) = \frac{1}{x^2}, \quad I_{1/x^2}(s,t) = \sqrt{st} = G(s,t),$$

$$f(x) = x^r, \quad r \neq 0, -1 \quad I_{x^r}(s,t) = \left(\frac{t^{r+1} - s^{r+1}}{(r+1)(t-s)}\right)^{\frac{1}{r}} = L_r(s,t),$$

* Corresponding author.
E-mail addresses: neven.elez@fer.hr (N. Elezović), lenka.vuksic@fer.hr (L. Vukšić).

http://dx.doi.org/10.1016/j.amc.2014.02.026
0096-3003/

where A, I, L, G, and L_r are arithmetic, identric, logarithmic, geometric and generalized logarithmic mean.

Although integral means are defined as functions of x, s and t, it turns out that the easier notation will be obtained if we introduce the following two variables:

$$\alpha = \frac{t+s}{2}, \quad \beta = \frac{t-s}{2}.$$

Then $t = \alpha + \beta$ and $s = \alpha - \beta$. All asymptotic expansions will be given in terms of α and β.

In recent papers [9,10] asymptotic expansions of many bivariate means are found, using the explicit formulas for observed means. The coefficients of asymptotic expansions are very useful in analysis of considered means. Here is a short table. General form of the expansion for a mean $M(s,t)$ is:

$$M(x+s, x+t) \sim x + \alpha + \sum_{n=2}^{\infty} c_n x^{-n+1}$$

and the first few coefficients are

Mean	c_2	c_3	c_4
A	0	0	0
I	$-\frac{1}{6}\beta^2$	$\frac{1}{6}\alpha\beta^2$	$-\frac{1}{360}\beta^2(60\alpha^2+13\beta^2)$
L	$-\frac{1}{3}\beta^2$	$\frac{1}{3}\alpha\beta^2$	$-\frac{1}{45}\beta^2(15\alpha^2+4\beta^2)$
G	$-\frac{1}{2}\beta^2$	$\frac{1}{2}\alpha\beta^2$	$-\frac{1}{8}\beta^2(4\alpha^2+\beta^2)$
L_r	$\frac{1}{6}(r-1)\beta^2$	$-\frac{1}{6}(r-1)\alpha\beta^2$	$\frac{1}{360}(r-1)\beta^2((-2r^2-5r+13)\beta^2+60\alpha^2)$

In the paper [7] the connection between differential and integral f-mean of a function f was analyzed and results were applied to digamma function. Among others, it was proved that

$$\psi\left(\frac{t-s}{\log t - \log s}\right) \leqslant \frac{1}{t-s}\int_s^t \psi(u)\,du,$$

i.e.,

$$L(s,t) \leqslant I_\psi(s,t).$$

Here, ψ denotes digamma function. As a consequence it follows that the function $x \mapsto I_\psi(x+s, x+t) - x$ is increasing and concave function and

$$I_\psi(x+s, x+t) - x \to A(s,t), \quad \text{as } x \to \infty.$$

This equation was essential in the proving the improvement of the lower bound in the second Gautschi–Kershaw inequalities, see [8, Theorem 4] for details:

$$\exp[\psi(x + I_\psi(s,t))] < \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} < \exp[\psi(x + A(s,t))].$$

In a recent paper [5] a complete asymptotic expansion of the function G in the formula

$$\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} = \exp[\psi(G(x))] \tag{1.2}$$

was found. The first few terms are

$$G(x) \sim c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \frac{c_4}{x^4} + \cdots, \tag{1.3}$$

where

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \alpha, \\ c_2 &= -\tfrac{\beta^2}{6}, \\ c_3 &= \tfrac{1}{12}\beta^2(2\alpha-1), \\ c_4 &= -\tfrac{1}{360}\beta^2\left[60\alpha^2 - 60\alpha + 13\beta^2 + 5\right]. \end{aligned} \tag{1.4}$$

It is easy to see that the form (1.2) is equivalent to

$$G(x) = I_\psi(x+s, x+t).$$