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A new Ostrowski–Grüss inequality involving $3n$ knots

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ABSTRACT

This is the fifth and last in our series of notes concerning some classical inequalities such as the Ostrowski, Simpson, Iyengar, and Ostrowski–Grüss inequalities in $\mathbb{R}$. In the last note, we propose an improvement of the Ostrowski–Grüss inequality which involves $3n$ knots where $n \geqq 1$ is an arbitrary numbers. More precisely, suppose that $\{x_k\}_{k=1}^n \subset [0,1]$, $\{y_k\}_{k=1}^n \subset [0,1]$, and $\{\alpha_k\}_{k=1}^n \subset [0,n]$ are arbitrary sequences with $\sum_{k=1}^n \alpha_k = n$ and $\sum_{k=1}^n \alpha_k x_k = n/2$. The main result of the present paper is to estimate

$$\frac{1}{n}\sum_{k=1}^n \alpha_k f(a+(b-a)y_k) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{f(b)-f(a)}{n}\sum_{k=1}^n \alpha_k (y_k - x_k)$$

in terms of either f' or f''. Unlike the standard Ostrowski–Grüss inequality and its known variants which basically estimate $f(x) - \left(\int_a^b f(t)\mathrm{d}t\right)/(b-a)$ in terms of a correction term as a linear polynomial of x and some derivatives of f, our estimate allows us to freely replace $f(x)$ and the correction term by using $3n$ knots $\{x_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$ and $\{\alpha_k\}_{k=1}^n$. As far as we know, this is the first result involving the Ostrowski–Grüss inequality with three sequences of parameters.

1. Introduction

It is no doubt that one of the most fundamental concepts in mathematics is inequality. However, as mentioned in a recent notes by Qi [23], the development of mathematical inequality theory before 1930 are scattered, dispersive, and unsystematic. Loosely speaking, the theory of mathematical inequalities has just formally started since the presence of a book by Hardy et al. [7]. Since then, the theory of mathematical inequalities has been pushed forward rapidly as a lot of books for inequalities were published worldwide.

Although the set of mathematical inequalities nowadays is huge, inequalities involving integrals and derivatives for real functions always have their own interest. Within this kind of inequalities, the one involving estimates of $\int_a^b f(t)\mathrm{d}t$ by bounds of the derivative of its integrand turns out to be fundamental as it has a long history and has received considerable attention from many mathematicians.

Not long before 1934, at the very beginning of the history of mathematical inequalities, in 1921, Pólya derived an inequality which can be used to estimate the integral $\int_a^b f(t)\mathrm{d}t$ by bounds of the first order derivative f'. His inequality basically says that the following holds

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$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqq \frac{b-a}{4}\|f'\|_\infty, \tag{1.1}$$

for any differentiable function f having $f(a) = f(b) = 0$ and $\|f'\|_\infty = \sup_{x\in[a,b]}|f'|$. Later on, in 1938, Iyengar [15] generalized (1.1) by showing that

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \leqq \frac{b-a}{4}\|f'\|_\infty - \frac{(f(b)-f(a))^2}{4(b-a)\|f'\|_\infty} \tag{1.2}$$

for any differentiable function f. Here the only difference is that the condition $f(a) = f(b) = 0$ is no longer assumed in (1.2). Apparently, (1.2) provides a simple error estimate for the so-called trapezoidal rule.

Also in this year, Ostrowski [21, page 226] proved another type of the Pólya–Iyengar inequality (1.2) which tells us how to approximate the difference $f(x) - \left(\int_a^b f(t)\mathrm{d}t\right)/(b-a)$ for $x \in [a,b]$. More precisely, he proved that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqq \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty \tag{1.3}$$

for all $x \in [a,b]$. As we have just mentioned, unlike (1.1), the inequality (1.3) provides a bound for the approximation of the integral average $\left(\int_a^b f(t)\mathrm{d}t\right)/(b-a)$ by the value $f(x)$ at the point $x \in [a,b]$.

Similar to the inequality (1.2), the Simpson inequality, which gives an error bound for the well-known Simpson rule, has been considered widely which is given as follows

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leqq \frac{\Gamma-\gamma}{12}(b-a), \tag{1.4}$$

where Γ and γ are real numbers such that $\gamma < f'(x) < \Gamma$ for all $x \in [a,b]$.

In recent years, a number of authors have written about generalizations of (1.1)–(1.4). For example, this topic is considered in [2,3,5,14,16,17,20,19,22,26,29]. In this way, some new types of inequalities are formed, such as inequalities of Ostrowski–Grüss type, inequalities of Ostrowski–Chebyshev type, etc.

The present paper is organized as the following. First, still in Section 1, let us use some space of the paper to mention several typical generalizations of (1.1)–(1.4). Later on, we shall review our recent works considering as generalizations of (1.1)–(1.4) which aims to propose a completely new idea in order to generalize these inequalities. In the final part of this section, we state our main result of the present paper whose proof is in Section 2.

1.1. Generalization of the Ostrowski inequality (1.3)

In the literature, there are several ways to generalize the Ostrowski inequality (1.3).

The first and most standard way is to replace the term $\|f'\|_\infty$ on the right hand side of (1.3) by $\|f'\|_q$ for any $q \geqq 1$ where, throughout the paper, we denote

$$\|g\|_q = \left(\int_a^b |g(t)|^q \mathrm{d}t\right)^{1/q},$$

for any function g. Within this direction, Theorem 1.2 in a monograph by Dragomir and Rassias [4] is the best as they were able to derive the best constant, see also [12, Theorem 2]. To be completed, let us recall the inequality that they proved

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqq \frac{(b-a)^{1/p}}{(p+1)^{1/p}}\left(\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right)^{1/p}\|f'\|_q$$

with $1/p + 1/q = 1$.

The second way to generalize the Ostrowski inequality (1.3) is to consider the so-called Ostrowski–Grüss type inequality. The only difference is that the term $(x - \frac{a+b}{2})\frac{f(b)-f(a)}{b-a}$ will be added to control $f(x) - \left(\int_a^b f(t)\mathrm{d}t\right)/(b-a)$. Within this type of generalization, let us recall a result due to Dragomir and Wang in [5, Theorem 2.1]. More precisely, they proved the following

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqq \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.5}$$

for all $x \in [a,b]$ where f' is integrable on $[a,b]$ and $\gamma \leqq f'(x) \leqq \Gamma$, for all $x \in [a,b]$ and for some constants $\gamma, \Gamma \in \mathbb{R}$.