Applied Mathematics and Computation 235 (2014) 318–322

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on the generalization of parameterized inexact Uzawa method for singular saddle point problems

Yuan Chen, Naimin Zhang *,[1]

School of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, People's Republic of China

ARTICLE INFO

Keywords:
Singular linear systems
Saddle point problems
Parameterized inexact Uzawa method
Semi-convergence

ABSTRACT

Recently, Zhang and Wang studied the generalized parameterized inexact Uzawa methods (GPIU) for solving singular saddle point problems (Zhang and Wang, 2013 [22]). In this note, we continue to discuss the semi-convergence of GPIU for solving singular saddle point problems, and weaken some semi-convergent conditions.

1. Introduction

We consider the iterative solution of a linear system with 2×2 block structure:

$$\mathcal{A}\mathcal{X} \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ a rank-deficient matrix, and $f \in R^m$ and $g \in R^n$ are given vectors, with $m \geqslant n$. We use B^T to denote the transpose of the matrix B. Linear system (1) is often called a saddle point problem, which arises in many application areas, such as computational fluid dynamics, mixed finite element approximation of elliptic partial differential equations, optimization, optimal control, constrained least-squares problems, electronic networks, computer graphics and others; see, e.g., [1,3,10,11,14,15,18,21] and references therein. When the coefficient matrix of linear system (1) is nonsingular, which requires B to be of full rank, a number of iterative methods have been proposed in the literature. For example, Uzawa-type methods which include parameterized Uzawa (PU) method and parameterized inexact Uzawa (PIU) method [6,7,13,16], Hermitian and skew-Hermitian splitting (HSS) methods [3,4], and a lot of preconditioned Krylov subspace iterative methods.

In this note, since the matrix B in (1) is rank-deficient, the coefficient matrix of (1) is singular, and (1) is called a singular saddle point problem. Recently, many techniques have been proposed for solving singular saddle point problems, including preconditioned minimum residual (PMINRES) method [17], preconditioned conjugate gradient (PCG) method [21]. For conjugate gradient type methods, here we also cite Restrictively preconditioned conjugate gradient (RPCG) methods [5,8]. Since Bai et al. proposed the PU and PIU methods [6,7], some authors developed these methods and used them to solve singular saddle point problems. Zheng et al. [24] applied the PU method to solve singular saddle point problems. Chen and Jiang [13] extended these methods and proposed a class of generalized inexact parameterized Uzawa methods. Ma and Zhang [20] studied block-diagonally preconditioned parameterized inexact Uzawa methods for singular saddle point problem.

* Corresponding author.
E-mail address: nmzhang@wzu.edu.cn (N. Zhang).

[1] This author is supported by Zhejiang Provincial Natural Science Foundation of China under grant No. Y1110451 and National Natural Science Foundation of China under grant No. 61002039.

http://dx.doi.org/10.1016/j.amc.2014.02.089
0096-3003/

Recently, Zhang and Wang [22] further studied the generalized parameterized inexact Uzawa (GPIU) methods for solving singular saddle point problems, and gave the corresponding semi-convergence analysis. In this note, we continue to discuss the GPIU methods for solving singular saddle point problems, and weaken some of the semi-convergence conditions in [22].

The rest of this note is organized as follow. In Section 2, we present the GPIU method for solving the singular saddle point problem (1), and give the corresponding semi-convergence analysis.

2. The semi-convergence of the GPIU method

For solving the singular saddle point problem (1), we make the following matrix splitting

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^T & \mathbf{0} \end{pmatrix} = \mathcal{M} - \mathcal{N}$$

where

$$\mathcal{M} = \begin{pmatrix} P & \mathbf{0} \\ -B^T + Q_1 & Q_2 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} P - A & -B \\ Q_1 & Q_2 \end{pmatrix}$$

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices, and $Q_1 \in R^{n\times m}$ is an arbitrary matrix. Then we present the following generalized parameterized inexact Uzawa (GPIU) iteration method [22] for solving the singular saddle point problem (1):

$$\begin{pmatrix} P & \mathbf{0} \\ -B^T + Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} P - A & -B \\ Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} x^k \\ y^k \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix} \tag{2}$$

or in block form,

$$\begin{cases} x^{k+1} = x^k + P^{-1}(f - Ax^k - By^k) \\ y^{k+1} = y^k + Q_2^{-1}(B^T x^{k+1} + g) - Q_2^{-1}Q_1(x^{k+1} - x^k) \end{cases} \tag{3}$$

and the iteration matrix $\mathcal{T}$ is:

$$\mathcal{T} = \begin{pmatrix} P & \mathbf{0} \\ -B^T + Q_1 & Q_2 \end{pmatrix}^{-1}\begin{pmatrix} P - A & -B \\ Q_1 & Q_2 \end{pmatrix} = I - \mathcal{M}^{-1}\mathcal{A} \tag{4}$$

As $\mathcal{A}$ is singular, then $\mathcal{T}$ has eigenvalue 1, and the spectral radius of the iteration matrix $\mathcal{T}$, i.e., $\rho(\mathcal{T})$ cannot be small than 1. For the iteration matrix $\mathcal{T}$, we introduce its pseudo-spectral radius $\upsilon(\mathcal{T})$,

$$\upsilon(\mathcal{T}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\}$$

where $\sigma(\mathcal{T})$ is the set of eigenvalues of $\mathcal{T}$.

For a matrix $B \in R^{n\times n}$, the smallest nonnegative integer k such that $rank(B^k) = rank(B^{k+1})$ is called the index of B, and we denote it by $k = index(B)$. In fact, $index(B)$ is the size of the largest Jordan block corresponding to the zero eigenvalue of B.

We now discuss the conditions of semi-convergence for solving singular linear systems, which have been studied by several authors (cf. [2,12,23]).

Lemma 2.1 [9]. *The iterative method (2) is semi-convergent, if and only if* $index(I - \mathcal{T}) = 1$ *and* $\upsilon(\mathcal{T}) < 1$.

Lemma 2.2 [22]. *Let* A, P *and* Q_2 *be symmetric positive definite, and* B *be of column rank-deficient,* Q_1 *is an arbitrary matrix. Suppose that* λ *is an eigenvalue of the iteration matrix* $\mathcal{T}$ *and* $(u^T, v^T)^T \in C^{m+n}$ *is the corresponding eigenvector. Then* $\lambda = 1$ *if and only if* $u = 0$.

Lemma 2.3 [22]. *Let* A, P *and* Q_2 *be symmetric positive definite,* B *be of rank-deficient and* Q_1 *is an arbitrary matrix. Suppose that* $\lambda \neq 1$ *is an eigenvalue of the iteration matrix* $\mathcal{T}$ *and* $(u^T, v^T)^T \in C^{m+n}$ *is the corresponding eigenvector. Then* λ *satisfies the following quadratic equation*

$$\lambda^2 + \frac{\beta + \gamma - 2\alpha - \tau}{\alpha}\lambda + \frac{\alpha + \tau - \beta}{\alpha} = 0 \tag{5}$$

where

$$\alpha = \frac{u^*Pu}{u^*u} > 0, \quad \beta = \frac{u^*Au}{u^*u} > 0, \quad \gamma = \frac{u^*BQ_2^{-1}B^Tu}{u^*u} \geqslant 0, \quad \tau = \frac{u^*BQ_2^{-1}Q_1u}{u^*u}$$