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Some properties on the tensor product of graphs obtained by monogenic semigroups

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ABSTRACT

In Das et al. (2013) [8], a new graph $\Gamma(\mathcal{S}_M)$ on monogenic semigroups $\mathcal{S}_M$ (with zero) having elements $\{0, x, x^2, x^3, \ldots, x^n\}$ has been recently defined. The vertices are the non-zero elements $x, x^2, x^3, \ldots, x^n$ and, for $1 \leqslant i, j \leqslant n$, any two distinct vertices x^i and x^j are adjacent if $x^i x^j = 0$ in $\mathcal{S}_M$. As a continuing study, in Akgunes et al. (2014) [3], it has been investigated some well known indices (first Zagreb index, second Zagreb index, Randić index, geometric–arithmetic index, atom-bond connectivity index, Wiener index, Harary index, first and second Zagreb eccentricity indices, eccentric connectivity index, the degree distance) over $\Gamma(\mathcal{S}_M)$.

In the light of above references, our main aim in this paper is to extend these studies over $\Gamma(\mathcal{S}_M)$ to the tensor product. In detail, we will investigate the diameter, radius, girth, maximum and minimum degree, chromatic number, clique number and domination number for the tensor product of any two (not necessarily different) graphs $\Gamma(\mathcal{S}_M^1)$ and $\Gamma(\mathcal{S}_M^2)$.

1. Introduction and preliminaries

The base of the graph $\Gamma(\mathcal{S}_M)$ is actually zero-divisor graphs (cf. [8]). In fact, the history of studying zero-divisor graphs has began over commutative rings by the paper [7], and then it followed over commutative and noncommutative rings by some of the joint papers [4–6]. After that the same terminology converted to commutative and noncommutative semigroups [10,11].

In a recent study [8], the graph $\Gamma(\mathcal{S}_M)$ is defined by changing the adjacent rule of vertices and not destroying the main idea. Detailed, the authors considered a finite multiplicative monogenic semigroup with zero as the set

$$\mathcal{S}_M = \{0, x, x^2, x^3, \ldots, x^n\}. \tag{1}$$

Then, by following the definition given in [11], it has been obtained an undirected (zero-divisor) graph $\Gamma(\mathcal{S}_M)$ associated to $\mathcal{S}_M$ as in the following. The vertices of the graph are labeled by the nonzero zero-divisors (in other words, all nonzero element) of $\mathcal{S}_M$, and any two distinct vertices x^i and x^j, where $(1 \leqslant i, j \leqslant n)$ are connected by an edge in case $x^i x^j = 0$ with the rule

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$x^ix^j = x^{i+j} = 0$ if and only if $i + j \geqslant n + 1$. The fundamental spectral properties such as the diameter, girth, maximum and minimum degree, chromatic number, clique number, degree sequence, irregularity index and dominating number for this new graph are presented in [8]. Furthermore, in [3], it has been studied first and second Zagreb indices, Randić index, geometric–arithmetic index and atom-bond connectivity index, Wiener index, Harary index, first and second Zagreb eccentricity indices, eccentric connectivity index and the degree distance to indicate the importance of the graph $\Gamma(\mathcal{S}_M)$.

It is known that studying the *extension* of graphs is also an important tool (see, for instance, [17,18]) since there are so many applications in sciences. With this idea, it is defined the *tensor product* $G_1 \otimes G_2$ of any two simple graphs G_1 and G_2 (in some references, it is also called *Kronecker product* [21]) which has the vertex set $V(G_1) \times V(G_2)$ such that any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are connected to each other by an edge if and only if $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. We may refer, for instance, [9,13,16,19,22] on some studies over tensor products of simple graphs. In this our study, we will replace G_1 by the graph $\Gamma(\mathcal{S}^1_M)$ and G_2 by the graph $\Gamma(\mathcal{S}^2_M)$, where $\mathcal{S}^1_M = \{x_1, x_1^2, x_1^3, \ldots, x_1^n\}$ with 0 and $\mathcal{S}^2_M = \{x_2, x_2^2, x_2^3, \ldots, x_2^m\}$ with 0 such that $n \geqslant m > 3$. Hence, the tensor product $\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)$ has a vertex set $V(\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)) = V(\Gamma(\mathcal{S}^1_M)) \times V(\Gamma(\mathcal{S}^2_M))$ which is given by

$$\left.\begin{array}{ll} \{(x_1, x_2), (x_1^2, x_2), \ldots, (x_1^n, x_2), & (x_1, x_2^2), (x_1^2, x_2^2), \ldots, (x_1^n, x_2^2), \\ \vdots & \vdots \\ (x_1, x_2^{m-1}), (x_1^2, x_2^{m-1}), \ldots, (x_1^n, x_2^{m-1}), & (x_1, x_2^m), (x_1^2, x_2^m), \ldots, (x_1^n, x_2^m)\} \end{array}\right\}. \tag{2}$$

Here, any two vertices (x_1^i, x_2^j) and (x_1^a, x_2^b) are connected to each other if and only if

$$\left.\begin{array}{l} x_1^ix_1^a \in E(\Gamma(\mathcal{S}^1_M)) \Longleftrightarrow x_1^ix_1^a = 0 \Longleftrightarrow i + a \geqslant n + 1 \\ \text{and} \\ x_2^jx_2^b \in E(\Gamma(\mathcal{S}^2_M)) \Longleftrightarrow x_2^jx_2^b \Longleftrightarrow j + b \geqslant m + 1 \end{array}\right\}. \tag{3}$$

In this paper, by considering $\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)$, we will present some certain results for the diameter, radius, girth, maximum and minimum degree, and finally chromatic, clique and domination numbers.

2. Main results

It is known that the *girth* of a simple graph G is the length of a shortest cycle contained in that graph. However, if G does not contain any cycle, then the girth of it is assumed to be infinity. Thus the first theorem of this paper is the following.

Theorem 2.1

$$Girth(\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)) = 3.$$

Proof. By considering (3), we easily see that the equalities

(*i*) $x_1^nx_1^{n-1} = 0$ and $x_2^mx_2^{m-1} = 0$ imply $(x_1^n, x_2^m) \sim (x_1^{n-1}, x_2^{m-1})$,
(*ii*) $x_1^{n-1}x_1^2 = 0$ and $x_2^{m-1}x_2^2 = 0$ imply $(x_1^{n-1}, x_2^{m-1}) \sim (x_1^2, x_2^2)$,
(*iii*) $x_1^2x_1^n = 0$ and $x_2^2x_2^m = 0$ imply $(x_1^2, x_2^2) \sim (x_1^n, x_2^m)$.

Then, thinking above steps at the same time, we get

$$(x_1^n, x_2^m) \sim (x_1^{n-1}, x_2^{m-1}) \sim (x_1^2, x_2^2) \sim (x_1^n, x_2^m),$$

as desired. □

The degree $deg_G(v)$ of a vertex v of G is the number of vertices which adjacent to v. Among all degrees, the *maximum* $\Delta(G)$ (or the *minimum* $\delta(G)$) degrees of G is the number of the largest (or the smallest) degree in G [14].

Theorem 2.2. *The maximum and minimum degrees of* $\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)$ *are*

$$\Delta(\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)) = (n-1)(m-1) \quad \text{and} \quad \delta(\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M)) = 1,$$

respectively.

Proof. Clearly, the vertex set $V(\Gamma(\mathcal{S}^1_M) \otimes \Gamma(\mathcal{S}^2_M))$ in (2) has total nm vertices. Among these vertices, let us take the vertex (x_1^n, x_2^m). The idea of (3) implies that it is not adjacent to the vertices

$$(x_1, x_2^m), (x_1^2, x_2^m), \ldots, (x_1^{n-1}, x_2^m) \quad \text{and} \quad (x_1^n, x_2), (x_1^n, x_2^2), \ldots, (x_1^n, x_2^{m-1}). \tag{4}$$