Applied Mathematics and Computation 235 (2014) 383–393

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An efficient derivative free family of fourth order methods for solving systems of nonlinear equations

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ARTICLE INFO

Keywords:
Systems of nonlinear equations
Derivative-free methods
Traub–Steffensen method
Order of convergence
Computational efficiency

ABSTRACT

We present a derivative free two-step family of fourth order methods for solving systems of nonlinear equations using the well-known Traub–Steffensen method in the first step. In order to determine the local convergence order, we apply the first-order divided difference operator for functions of several variables and direct computation by Taylor's expansion. Computational efficiencies of the methods of new family are considered and compared with existing methods of similar structure. It is showed that the new family is especially efficient in solving large systems. Four numerical examples are given to compare the proposed methods with existing methods and to confirm the theoretical results.

1. Introduction

The construction of fixed point iterative methods for solving nonlinear equations or systems of nonlinear equations is an interesting and challenging task in numerical analysis and many applied scientific disciplines. The importance of this subject has led to the development of many numerical methods, most frequently of iterative nature. In the past few decades, iterative techniques have been applied in many diverse fields as economics, engineering, physics, dynamical models, and so on.

In this paper we consider the problem of finding solution of the system of nonlinear equations $F(x) = 0$ by iterative methods of fourth order, where $F : D \to D, D$ is an open convex domain in R^m and x is a vector of m unknowns. One of the best known iterative procedures for solving nonlinear equations is the quadratically convergent Newton's method (see [1,2]),

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \quad k = 0, 1, 2, \ldots,$$

where $[F'(x)]^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$.

In many practical applications it is too complicated to calculate the derivative $F'(x)$ of the function $F(x)$ of m variables. In such situations it is preferable to use only the computed values of $F(x)$ and to approximate $F'(x)$ by employing the values of $F(x)$ at suitable points. For example, a basic quadratically convergent derivative free iterative method is the Traub–Steffensen method [3],

$$x^{(k+1)} = M_{2,1}(x^{(k)}) = x^{(k)} - [F; u^{(k)}, x^{(k)}]^{-1}F(x^{(k)}), \tag{1}$$

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http://dx.doi.org/10.1016/j.amc.2014.02.103
0096-3003/

where $[F;u^{(k)},x^{(k)}]^{-1}$ is the inverse of the first order divided difference $[F;u^{(k)},x^{(k)}]$ of F and $u^{(k)}=x^{(k)}+bF(x^{(k)})$, b is an arbitrary non-zero constant. Here and in the sequel, $\mathrm{M}_{p,i}$ is used for denoting ith method of convergence order p. For $b=1$ the iterative scheme (1) reduces to the method of Steffensen's type [4].

The design of efficient higher order derivative free methods for systems of nonlinear equations of the form $F(x)=0$ is a very difficult task and so only a few efficient methods have been developed at present in spite of the fact that many derivative free higher order methods have been proposed in the literature, see, e.g., [5–14] and references therein. More details about techniques for construction of derivative free methods for nonlinear systems can be found, e.g. in [13,15–17].

Based on Steffensen's type scheme, Wang et al. [13] have generalized the method of Ren et al. [5] for scalar equations to systems of nonlinear equations, which is given by

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}),\\ x^{(k+1)} &= \mathrm{M}_{4,1}(x^{(k)},y^{(k)}) = y^{(k)} - ([F;y^{(k)},x^{(k)}]+[F;y^{(k)},u^{(k)}]-[F;u^{(k)},x^{(k)}])^{-1}F(y^{(k)}). \end{aligned}$$

Wang et al. have also extended the method by Liu et al. [6] to solve systems of nonlinear equations in [13], which is given as

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}),\\ z^{(k)} &= \mathrm{M}_{4,2}(x^{(k)},y^{(k)}) = y^{(k)} - [F;y^{(k)},x^{(k)}]^{-1}([F;y^{(k)},x^{(k)}]-[F;y^{(k)},u^{(k)}]+[F;u^{(k)},x^{(k)}])[y^{(k)},x^{(k)};F]^{-1}F(y^{(k)}). \end{aligned}$$

Recently, based on Traub–Steffensen iteration (1), the following fourth order two-step method has been proposed in [17],

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}),\\ x^{(k+1)} &= \mathrm{M}_{4,3}(x^{(k)},y^{(k)}) = y^{(k)} - (3I-[F;u^{(k)},x^{(k)}]^{-1}([F;y^{(k)},x^{(k)}]+[F;y^{(k)},u^{(k)}]))[F;u^{(k)},x^{(k)}]^{-1}F(y^{(k)}). \end{aligned}$$

The main goal of this paper is to develop an efficient derivative free family of two-step methods, which assumes fourth convergence order and low computational costs. To do this, we propose a family of methods with fourth order of convergence by employing an iterative scheme that utilizes as minimum number of function evaluations as possible. In this way we attain low computational cost and hence an increased computational efficiency. Moreover, we show that the proposed methods are efficient than existing derivative free methods in general.

We summarize contents of the paper. In Section 2, a new fourth order two-step family is developed and its convergence analysis is presented. In Section 3, the computational efficiency of the new family is discussed and compared with methods of similar structure. Various numerical examples are considered in Section 4 to show the consistent convergence behavior of the proposed methods and to verify the theoretical results. Section 5 contains concluding remarks.

2. Development of the method

In numerical analysis an iterative root-finding method is regarded as computationally efficient if it attains a high computational speed with minimum computational cost. A most obvious barrier in the development of an efficient iterative scheme for solving systems of nonlinear equations is the evaluation of inverse of a matrix since it is very expensive from a computational point of view. Therefore, when constructing a numerical algorithm it will be more appropriate if the number of matrix inversions is as small as possible. Keeping this fact in mind we start from the following scheme:

$$\begin{aligned} y^{(k)} &= \mathrm{M}_{2,1}(x^{(k)}),\\ x^{(k+1)} &= y^{(k)} - (\alpha I + G^{(k)}(\beta I + \gamma G^{(k)}))[F;u^{(k)},x^{(k)}]^{-1}F(y^{(k)}), \end{aligned} \tag{2}$$

where $\mathrm{M}_{2,1}(x^{(k)})$ is the iterative scheme defined by (1), $G^{(k)}=[F;u^{(k)},x^{(k)}]^{-1}[F;z^{(k)},y^{(k)}]$, $z^{(k)}=y^{(k)}+cF(y^{(k)})$, I is the identity matrix, c is an arbitrary non-zero constant and α,β,γ are some parameters to be determined.

In order to explore the convergence property of (2), we recall the following result of Taylor's expansion on vector functions (see [2]).

Lemma 1. *Let $F:D\subset R^m\to R^m$ be p-times Fréchet differentiable in a convex set $D\subset R^m$ then for any $x,h\in R^m$, the following expression holds:*

$$F(x+h)=F(x)+F'(x)h+\frac{1}{2!}F''(x)h^2+\frac{1}{3!}F'''(x)h^3+\cdots+\frac{1}{p!}F^{(p-1)}(x)h^{p-1}+R_p, \tag{3}$$

where

$$\|R_p\|\leqslant\frac{1}{p!}\sup_{0\leqslant t\leqslant 1}\|F^{(p)}(x+th)\|\,\|h\|^p \quad \textit{and} \quad h^p=(h,h,\overset{p}{\cdots},h).$$

Let us now recall the definition of divided difference operator for multivariable function F (see [18]). The divided difference operator of F is a mapping $[F;\cdot,\cdot]:D\times D\subset R^m\times R^m\to L(R^m)$ defined by

$$[F;x+h,x]=\int_0^1 F'(x+th)\,dt, \quad \forall x,h\in R^m. \tag{4}$$