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The eigenvalue for a class of singular p-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition ☆

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ABSTRACT

In this paper, we are concerned with the eigenvalue problem of a class of singular p-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition. The conditions for the existence of at least one positive solution is established together with the estimates of the lower and upper bounds of the solution at any instant of time. Our results are derived based on the method of upper and lower solutions and the Schauder fixed point theorem.

1. Introduction

This paper deals with the eigenvalue problem for a class of singular p-Laplacian fractional differential equations (PFDE for short) involving the Riemann–Stieltjes integral boundary condition

$$\begin{cases} -\mathscr{D}_t^{\beta}\left(\varphi_p(\mathscr{D}_t{}^{\alpha}x)\right)(t) = \lambda f(t,x(t)), \quad t \in (0,1), \\ x(0)=0, \quad \mathscr{D}_t^{\alpha}x(0)=0, \quad x(1)=\int_0^1 x(s)dA(s), \end{cases} \tag{1.1}$$

where $\mathscr{D}_t^{\beta}$ and $\mathscr{D}_t^{\alpha}$ are the standard Riemann–Liouville derivatives with $1<\alpha\leqslant 2,\ 0<\beta\leqslant 1$. A is a function of bounded variation and $\int_0^1 x(s)dA(s)$ denotes the Riemann–Stieltjes integral of x with respect to A, the p-Laplacian operator φ_p is defined as $\varphi_p(s)=|s|^{p-2}s,\ p>1, f(t,x):(0,1)\times(0,+\infty)\to[0,+\infty)$ is continuous and may be singular at $t=0,1$ and $x=0$.

Integral and derivative operators of fractional order can describe the characteristics exhibited in many complex processes and systems having long-memory in time, and for this reason many classical integer-order models for complex systems are being substituted by fractional order models. Fractional calculus also provides an excellent tool to describe the hereditary properties of materials and processes, particularly in viscoelasticity, electrochemistry and porous media (see [1–5]). Many successful new applications of fractional calculus in various fields have also been reported recently. For example, Nieto and Pimentel [18] extended a second-order thermostat model to the fractional model, Ding and Jiang [19] used waveform

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relaxation methods to study some fractional functional differential equations models. For the basic theories of fractional calculus and some recent work in application, the reader is referred to Refs. [17,23–34].

Much of the work on fractional calculus deals with boundary value problems [6–9,20]. In particular, Ahmad and Nieto [20] considered a nonlinear Dirichlet boundary value problem of sequential fractional integro-differential equations in the sense of the Caputo fractional derivative, and the existence results are established by means of some standard tools of fixed point theory. On the other hand, some developments on the topic involving the p-Laplacian operator and complex boundary value conditions have been reported [10–13,21]. In [22], Li and Lin considered a Hadamard fractional boundary value problem with a p-Laplacian operator as below:

$$\begin{cases} \mathscr{D}_t^\beta\left(\varphi_p(\mathscr{D}_t{}^\alpha x)\right)(t) = f(t,x(t)), \quad 1 < t < e, \\ x(1) = x'(1) = x'(e) = 0, \quad \mathscr{D}_t^\alpha x(1) = \mathscr{D}_t^\alpha x(e) = 0, \end{cases}$$

where $2 < \alpha \leqslant 3,\ 1 < \beta \leq 2, \varphi_p(s) = |s|^{p-2}s,\ p > 1$, and $f : [1,e] \times [0,+\infty) \to [0,+\infty)$ is a positive continuous function. By using the Leray–Schauder type alternative and the Guo–Krasnoselskii fixed point theorem, the existence and the uniqueness of the positive solutions were established.

The aim of this paper is to deals with the eigenvalue problem for the PFDE involving the Riemann–Stieltjes integral boundary condition, which allows the boundary conditions to be quite general, by using the upper and lower solutions and the Schauder fixed point theorem, so as to determine the interval of eigenvalue for the existence of positive solutions.

2. Preliminaries and lemmas

Denote by $C[0,1]$ the space of all continuous functions on $[0,1]$ with the usual norm $||x|| = \max_{0\leqslant t\leqslant 1}|x(t)|$. Indeed, $C[0,1]$ is a Banach space with a partial order, namely for $x, y \in C[0,1],\ x \leqslant y \Longleftrightarrow x(t) \leqslant y(t)$, for $t \in [0,1]$.

Now consider the linear fractional differential equation

$$\begin{cases} \mathscr{D}_t^\alpha x(t) + h(t) = 0, \quad t \in (0,1), \\ x(0) = 0, \quad x(1) = \int_0^1 x(s)dA(s). \end{cases} \tag{2.1}$$

Lemma 2.1 [14]. *Given $h(t) \in L^1[0,1]$, the problem*

$$\begin{cases} \mathscr{D}_t^\alpha x(t) + h(t) = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = 0 \end{cases} \tag{2.2}$$

has the unique solution

$$x(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.3}$$

where $G(t,s)$ is the Green function of (2.2) and is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} [t(1-s)]^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \\ [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.4}$$

Consider the problem

$$\begin{cases} \mathscr{D}_t^\alpha x(t) = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = 1, \end{cases} \tag{2.5}$$

by the property of the Riemann–Liouville fractional integral and derivative operators, the unique solution of the problem (2.5) is $t^{\alpha-1}$. Defining $\mathcal{G}_A(s) = \int_0^1 G(t,s)dA(t)$, as in [15,16], we can get that the Green function for the nonlocal FDE (2.1) is given by

$$J(t,s) = \frac{t^{\alpha-1}}{1-\mathcal{C}}\mathcal{G}_A(s) + G(t,s), \quad \mathcal{C} = \int_0^1 t^{\alpha-1}dA(t). \tag{2.6}$$

Lemma 2.2 [16]. *Let $0 \leqslant \mathcal{C} < 1$ and $\mathcal{G}_A(s) \geq 0$ for $s \in [0,1]$, then the Green function defined by (2.6) satisfies:*

(1) *$J(t,s) > 0$, for all $t, s \in (0,1)$.*
(2) *There exist two constants c_*, c^* such that*

$$c_* t^{\alpha-1}\mathcal{G}_A(s) \leqslant J(t,s) \leqslant c^* t^{\alpha-1} \leqslant c^*, \quad t,s \in [0,1], \tag{2.7}$$