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Oscillation criteria for nth order nonlinear neutral differential equations ☆

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ABSTRACT

Sufficient conditions are established for the oscillation of nth order neutral differential equations of the form

$$\left(r(t)\left(x(t)|x(t)|^{\alpha-1}+p(t)x(\tau(t))\right)^{(n-1)}\right)'+q(t)f(x(\sigma(t)))=0,\quad t\geqslant t_0,$$

where $n\geqslant 2$ is even integer, $\alpha\geqslant 1,\ p,q\in C([t_0,+\infty),R),\ f\in C(R,R)$. The results obtained extend some of the known results.

1. Introduction

In this paper, we are concerned with the oscillation behavior of solution of the nth order neutral differential equations of the form

$$\left(r(t)\left(x(t)|x(t)|^{\alpha-1}+p(t)x(\tau(t))\right)^{(n-1)}\right)'+q(t)f(x(\sigma(t)))=0,\quad t\geqslant t_0,\tag{1.1}$$

where $n\geqslant 2$ is even integer, $\alpha\geqslant 1,\ p,q\in C([t_0,+\infty),R),\ f\in C(R,R)$. we assume that

(H_1) $0\leqslant p(t)\leqslant 1,\ q(t)\geqslant 0$;
(H_2) $r\in C'([t_0,+\infty),(0,+\infty)),\ r(t)>0,\ r'(t)\geqslant 0,\ \int_{t_0}^{+\infty}\frac{1}{r(t)}dt=\infty$;
(H_3) $\frac{f(x)}{|x|^{\alpha-1}x}\geqslant\beta>0$, for $x\neq 0$, β is a constant;
(H_4) $\tau,\sigma\in C([t_0,+\infty),[0,+\infty)),\ \tau(t)\leqslant t,\ \lim_{t\to\infty}\tau(t)=\infty$;
(H_5) $\sigma\in C'([t_0,+\infty),(0,+\infty)),\ \sigma(t)\leqslant t,\ \sigma'(t)>0,\ \lim_{t\to\infty}\sigma(t)=\infty$.

By a solution of Eq. (1.1), we mean a function $x(t)\in C([t_k,+\infty),R)$ for some $t_k\geqslant t_0$, such that $x(t)|x(t)|^{\alpha-1}+p(t)x(\tau(t))\in C^n([t_k,+\infty),R)$ and $r(t)\left(x(t)|x(t)|^{\alpha-1}+p(t)x(\tau(t))\right)^{(n-1)}\in C'([t_k,+\infty),R)$ and satisfies Eq. (1.1) on $[t_k,+\infty)$. a nontrivial solution $x(t)$ of Eq. (1.1) is called oscillatory in $[t_0,+\infty),t_0>0$ if it has arbitrarily large zeros. Another word, a nontrivial solution $x(t)$ of Eq. (1.1) is called oscillatory if there exist a sequence of real numbers $\{t_k\}_{k=1}^{\infty}$, diverging to

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$+\infty$, such that $x(t_k)=0$. Otherwise the solution is called nonoscillatory. Neutral differential Eq. (1.1) is called be oscillatory if all its solutions are oscillatory.

We develop certain theorems related to the oscillatory behavior and provide sufficient conditions for the above equation to be oscillatory. The oscillatory behavior of neutral differential equation of nth order has been the subject of several papers [1–10]. Eq. (1.1) with $r(t)=1$, namely, the equation

$$\left(\left(x(t)|x(t)|^{\alpha-1}+p(t)x(\tau(t))\right)^{(n-1)}\right)'+q(t)f(x(\sigma(t)))=0,\quad t\geqslant t_0 \tag{1.2}$$

and related equations have been investigated by Dahiya and Candan [3], Candan and Dahiya [6,7]. Our result are general than those of [3,6,7].

If $n=2,\ \alpha=1$, then Eq. (1.1) becomes

$$(r(t)(x(t)+p(t)x(\tau(t)))')'+q(t)f(x(\sigma(t)))=0,\quad t\geqslant t_0 \tag{1.3}$$

and related equations have been studied by Ruan [11] and Li and Liu [12]. The purpose of this paper is to improve and extend above mentioned results. We shall further offer some new criteria for the oscillation of Eq. (1.1).

2. Main results

In order to prove our theorems we shall need the following three lemmas.

Lemma 2.1 [13]. *Let $y(t)$ be an n times differentiable function on $[t_0,+\infty)$ of constant sign, $y^{(n)}(t)\neq 0$ on $[t_0,+\infty)$ which satisfies $y^{(n)}(t)y(t)\leqslant 0$. Then*

- (I_1) *There exists $t_1\geqslant t_0$ such that the functions $y^{(i)}(t),\ i=1,2\ldots,n-1$ are of constant sign on $[t_1,+\infty)$;*
- (I_2) *There exists a number $l\in\{1,3,5,\ldots,n-1\}$ when n is even, or $l\in\{0,2,4,\ldots,n-1\}$ when n is odd, such that $y(t)y^{(i)}(t)>0$ for $i=0,1,\ldots,l,\ t\geqslant t_1$; $(-1)^{n+i+1}y(t)y^{(i)}(t)>0$ for $i=l+1,\ldots,n,\ t\geqslant t_1$.*

Lemma 2.2 [13]. *If the function $y(t)$ is as in Lemma 2.1 and $y^{(n-1)}(t)y^{(n)}(t)\leqslant 0$ for $t\geqslant t_0$, then for every $\lambda,\ 0<\lambda<1$, there exist a constant $M>0$ such that*

$$|y(\lambda t)|\geqslant Mt^{n-1}|y^{(n-1)}(t)|$$

for all large t.

Lemma 2.3. *Suppose that $x(t)$ is an eventually positive solution of Eq. (1.1), let*

$$z(t)=x(t)|x(t)|^{\alpha-1}+p(t)x(\tau(t)), \tag{2.1}$$

then there exists a number $t_1\geqslant t_0$ such that

$$z(t)>0,\quad z'(t)>0,\quad z^{(n-1)}(t)\geqslant 0,\quad z^{(n)}(t)\leqslant 0,\quad t\geqslant t_1. \tag{2.2}$$

Proof. Since $x(t)$ is an eventually positive solution of Eq. (1.1), there exists a number $t_1\geqslant t_0$ such that $x(t)>0,\ x(\tau(t))>0,\ x(\sigma(t))>0,\ t\geqslant t_1$. From (2.1), we have $z(t)>0,\ t\geqslant t_1$ and

$$(r(t)z^{(n-1)}(t))'=-q(t)f(x(\sigma(t)))\leqslant 0,\quad t\geqslant t_1.$$

It follows that the function $r(t)z^{(n-1)}(t)$ is decreasing and $z^{(n-1)}(t)$ is eventually of one sign. We claim that $z^{(n-1)}(t)\geqslant 0,\ t\geqslant t_1$. Otherwise, if there exist a $\bar{t}_0\geqslant t_1$ such that $z^{(n-1)}(\bar{t}_0)<0,\ t\geqslant \bar{t}_0$ and

$$r(t)z^{(n-1)}(t)\leqslant r(\bar{t}_0)z^{(n-1)}(\bar{t}_0)=-C\quad (C>0),$$

which implies that $r(t)z^{(n-1)}(t)\leqslant -C,\ t\geqslant \bar{t}_0$, that is $-z^{(n-1)}(t)\geqslant \frac{C}{r(t)}$, integrating the above inequality from $\bar{t}_0$ to t, we have

$$z^{(n-2)}(t)\leqslant z^{(n-2)}(\bar{t}_0)-C\int_{\bar{t}_0}^{t}\frac{1}{r(s)}ds.$$

Letting $t\to+\infty$, from (H_2), we get $\lim_{t\to\infty}z^{(n-2)}(t)=-\infty$, which implies $z(t)$ is eventually negative by Lemma 2.1. This is a contradiction. Hence $z^{(n-1)}(t)\geqslant 0,\ t\geqslant t_1$. Furthermore, from Eq. (1.1) and (H_2), we have

$$r(t)z^{(n)}(t)=-r'(t)z^{(n-1)}(t)-q(t)f(x(\sigma(t)))\leqslant 0,\quad t\geqslant t_1.$$

This imply that $z^{(n)}(t)\leqslant 0,\ t\geqslant t_1$. From Lemma 2.1 again (note n is even), we have $z'(t)>0,\ t\geqslant t_1$. This completes the proof. $\square$