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Differential systems for constrained optimization via a nonlinear augmented Lagrangian

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ABSTRACT

A general differential system framework for solving constrained optimization problems is investigated, which relies on a class of nonlinear augmented Lagrangians. The differential systems mainly consist of first-order derivatives and second-order derivatives of the nonlinear augmented Lagrangian. Under suitable conditions, the asymptotic stability of the differential systems and local convergence properties of their Euler discrete schemes are obtained, including the locally quadratic convergence rate of the discrete sequence for the second-order derivatives based differential system. Furthermore, as the special case, the exponential Lagrangian applied to this framework is given. Numerical experiments are presented illustrating the performance of the differential systems.

1. Introduction

Consider the following optimization problems:

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } g_i(x) \geqslant 0, \quad i = 1, \ldots, m. \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}(i = 1, \ldots, m)$ are twice continuously differentiable functions. It is well known that the augmented Lagrange methods has many applications in the study of optimization problems. The first augmented Lagrangian, namely the proximal Lagrangian, was introduced by Rockafellar [1] and the theory of augmented Lagrangians were developed in, e.g., Ioffe [2], Bertsekas [3,4] and Rockafellar [5] for constrained optimization problems. Originally, the method was applied to problems with equality constraints [6,7] and later generalized to problems with inequality constraints [8,9]. By using augmented Lagrangian, constrained optimization problems can be translated into unconstrained optimization problems.

The goal of this work is to apply differential equation methods for solving problem (1). The main idea of this type of methods is to construct the differential systems based on the nonlinear augmented Lagrangian. The corresponding systems mainly consist of first-order derivatives and second-order derivatives of the nonlinear augmented Lagrangian. It is demonstrated that the equilibrium point of this system coincides with the solution to the constrained optimization. The analysis of the methods is made on the basis of the stability theory of the solution of ordinary differential equations. In the original version of the differential equation methods was introduced by Arrow and Hurwicz [10], some results have been addressed in

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the work (see Refs. [11–13] for details). Among them, Evtushenko and Zhadan [14–18] have studied, by using the so-called space transformation techniques, a family of numerical methods for solving optimization problems with equality and inequality constraints. The proposed algorithms are based on the numerical integration of the systems of ordinary differential equation. Along this line, Zhang [19–21] studied modified versions of differential equation methods.

This paper develops the results obtained in Ref. [22,23] motivated by finding a unified approach to the construction a family of differential systems to solve inequality constrained problems without using space transformations of Evtushenko and Zhadan. The method are based on a class of nonlinear augmented Lagranians, by which we construct first-order derivatives based and second-order derivatives based differential systems. The differential systems are carried out without using the space transformation and this feature provides a high rate of convergence. Under a set of suitable conditions, we prove the asymptotical stability of the two class of differential systems and local convergence properties of their Euler discrete schemes, including the locally quadratic convergence rate of the discrete sequence for second order derivatives based differential equation systems. Under this framework, two specific eases, the differential equation systems generated by the exponential Lagranian and the modified barrier function have been discussed in Ref. [22,23].

The paper is organized as follows. A class of nonlinear augmented Lagrangian and related properties are introduced in Section 2.1. Section 2.2 derives the differential system based on the nonlinear augmented Lagrangian and the asymptotic stability theorem is established under mild conditions. In Section 2.3, the Euler discrete schemes for the differential system is presented and the local convergence theorem is demonstrated. Under this framework, two specific cases, the differential systems generated by the exponential Lagrangian and the modified barrier function, are proposed in Section 2.4. In Section 3.1, we construct a second-order derivatives based differential system and prove the asymptotic stability of the system. In Section 3.2, Euler discrete schemes and their local convergence properties are obtained, including the locally quadratic convergence rate of the discrete sequence for the second order derivatives based differential system. The numerical results show that Runge–Kutta method has better stability and higher precision in Section 4.

2. Differential systems based on first-order derivatives

In this section, we mention some preliminaries that will be used throughout this paper. Without loss of generality, we assume that $f(x)$ and $g_i(x),\ i=1,\dots,m$ are twice continuously differentiable, then the classical Lagrangian for problem (1) defined by $L(x,\lambda)=f(x)-\sum_{i=1}^{m}\lambda_i g_i(x)$, for any feasible point x, the active set of indices is denoted by $I(x)=\{i|g_i(x)=0,\ i=1,\dots,m\}$. Let x^* be a local minimizer to problem (1) and the pair $(x^*,\ \lambda^*)$ be the corresponding KKT point, which satisfies the following conditions:

$$\nabla_x L(x^*,\lambda^*)=\nabla f(x^*)-\sum_{i=1}^{m}\lambda_i^*\nabla g_i(x^*)=0,\quad \lambda_i^*\geqslant 0,\quad \lambda_i^* g_i(x^*)=0,\quad g_i(x^*)\geqslant 0,\quad i=1,\dots,m. \tag{2}$$

Let the Jacobian uniqueness conditions, proposed in [11], hold at (x^*,λ^*):

(1) The multipliers $\lambda^*>0,\ i\in I(x^*)$.
(2) The gradients $\nabla g_i(x^*),\ i\in I(x^*)$ are linearly independent.
(3) $y^T\nabla_{xx}^2 L(x^*,\ \lambda^*)y>0,\quad \forall 0\neq y\in\{y|\nabla g_i(x^*)^T y=0,\ i\in I(x^*)\}$.

2.1. The properties of nonlinear augmented Lagrangians

We introduce a class of nonlinear augmented Lagrangian for problem (1), in the following form

$$F(x,y,t)=f(x)-|t|\sum_{i=1}^{m}\Psi(|t|^{-1}g_i(x),y_i),\quad (t\neq 0),$$

where t is a controlling parameter, $\Psi(w,v):\mathbb{R}^2\to\mathbb{R}^1$ is a real-value function and $y_i(i=1,\dots,m)$ are variables of the nonlinear Lagrangian multiplier vector. By using nonlinear augmented Lagrangian $F(x,y,t)$, the differential systems for solving problem (1) will be derived in the next section.

The following lemma will be used in the proof of the forthcoming theorem.

Lemma 2.1 (*See Ref. [3]*). *Let A be a $n\times n$ symmetrical matrix, B be a $r\times n$ matrix, $U=[diag\mu_i]_{i=1}^{m}$, where $\mu=(\mu_1,\dots,\mu_r)>0$. If $\lambda>0$ is a scalar and $By=0$ implies $\langle Ay,y\rangle\geqslant\lambda\langle y,y\rangle$. Then there are scalars $k_0>0$ and $c\in(0,\lambda)$ such that, for any $k\geqslant k_0$,*

$$\langle (A+kB^TUB)x,x\rangle\geqslant c\langle x,x\rangle,\quad \forall x\in\mathbb{R}^n.$$

Now we discuss the properties of nonlinear augmented Lagrangian$F(x,y,t)$.

Theorem 2.1. *Let (x^*,λ^*) be a KKT point of (1), the Jacobian uniqueness conditions hold at (x^*,λ^*), and the following conclusions are satisfied:*