Applied Mathematics and Computation 235 (2014) 502–511

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Dependence of exchangeable residual lifetimes subject to failure

Fabrizio Durante [a], Rachele Foschi [b,*]

[a] School of Economics and Management, Free University of Bozen-Bolzano, Bolzano, Italy
[b] Laboratory of Innovation Management and Economics, IMT Advanced Studies, Lucca, Italy

ARTICLE INFO

Keywords:
Copula
Dependence
Lifetime
Survival analysis
Risk Models

ABSTRACT

We provide exact formulas about the survival copula of an exchangeable random vector of residual lifetimes when some of the components have failed. Specific computations are carried out in the case of Archimedean copulas and the evolution of dependence after successive failures of some components is shown numerically via measures of association.

1. Introduction

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a random vector on a suitable probability space representing the lifetimes of an engineering system composed by several components that are possibly subject to failure. Although the components are often assumed to be independent, this assumption may not be valid in practice, since a system is usually working in a common environment and suffers from the same stress. In order to introduce dependence in the lifetimes in a convenient way, the concept of copula has been recently used in various situations: see, for instance, [35,24,20,30,12–15,29,28] and the references therein.

The study of the dependence (e.g., copula-based) properties of $\mathbf{T}$ at the increase of age has been the object of several recent investigations; see, for instance, [21,22,5], which are mainly motivated by applications to risk management, and [3,32], which adopt the viewpoint of reliability analysis. In particular, in the case $n = 2$ and in absence of any failure in the system, such a topic has been investigated in [9].

Following the copula approach, our aim is to study the evolution of survival distributions among residual lifetimes of $\mathbf{T}$ when some of the vector components have failed. These investigations have been considered, for instance, in credit risk, where the main interest is in the default dependencies between obligors (see, for instance, [33,26]) and their impact on the survival probability of the remaining variables.

Here, instead, we suppose that at least two components of the system remains alive and we focus on the calculation of survival copulas at different residual lifetimes and in presence of failures of the remaining components. Such studies are hence related to the determination of the evolution of dependence properties of an $n - k + 1$ out of n system, i.e. a system that works successfully until k of the components have failed. In particular, compared to previous approaches that usually assume independence or conditional independence between the system lifetimes (see, e.g., [2]), we adopt a general form of dependence as described by the survival copula of the system. Moreover, notice that, the information about the copula at different times (together with the marginal behaviour) can be used to express and interpret some multivariate Bayesian notions of aging of the whole system [4,16,10].

* Corresponding author.
E-mail addresses: fabrizio.durante@unibz.it (F. Durante), rachele.foschi@imtlucca.it (R. Foschi).

http://dx.doi.org/10.1016/j.amc.2014.02.069
0096-3003/

The rest of the paper is organized as follows. In Section 2 we present the preliminaries about our model. Section 3 presents exact formulas about the survival copula of residual lifetimes with and without failures in the system. In order to illustrate the main results, specific computations are carried out in the case of Archimedean copulas (the so-called TTE models in [4]) and the evolution of dependence is shown numerically with the calculation of suitable measures of association. Section 4 concludes.

2. The survival model

We consider an exchangeable survival model of lifetime $\mathbf{T} = (T_1, \dots, T_n)$ such that each component is a non-negative random variable. Denote by $T_{1:n}, \dots, T_{n:n}$ the order statistics of $\mathbf{T}$. The vector $\mathbf{T}$ is determined by the joint survival function

$$\overline{F}(x_1, \dots, x_n) = \mathbb{P}(T_1 > x_1, \dots, T_n > x_n),$$

with survival margins $\overline{F}^{(J)}(x_1, \dots, x_r) = \mathbb{P}(T_{i_1} > x_1, \dots, T_{i_r} > x_r)$, where $J = \{i_1, \dots, i_r\}$ is a subset of $\{1, \dots, n\}$ with cardinality $r = |J|$. Notice that the exchangeability assumption is less restrictive than it could appear at a first glance. In fact, given any vector $\mathbf{T}$ of lifetimes it is always possible to trace back to an exchangeable vector $\mathbf{X}$ having the same order statistics of $\mathbf{T}$, and, hence, sharing similar properties in terms of lifetime of the system (for more details, see [18,36]).

We assume that $\overline{F}$ is absolutely continuous, with density f and univariate survival margin equal to $\overline{G}$ that is strictly decreasing in $(0, +\infty)$. As known (see e.g. [31,20,25]), the dependence properties of a survival function $\overline{F}$ can be described in terms of the corresponding survival copula K given, for all $\mathbf{u} \in [0,1]^n$, by

$$K(\mathbf{u}) = \overline{F}\Big(\overline{G}^{-1}(u_1), \dots, \overline{G}^{-1}(u_n)\Big). \tag{1}$$

Here our aim is to study the time evolution of the dependence among the components of $\mathbf{T}$, with special attention to the evolution of dependence at the instant of failure of one component. To this end, we introduce the following notation. For every $t > 0$ and $k = 0, 1, \dots, n-2$, we define the event $H_t^{t_1,\dots,t_k}$, which represents the history of the model at time t when the last k components of $\mathbf{T}$ have failed at known times $0 < t_1 < \dots < t_k$. In particular, in view of the exchangeability of the system, we suppose without loss of generality that the failed components are the k last ones and the failures affect first the last component, then the component of index $n-1$, and so on. We also suppose that, at any instant, at most one failure may occur.

We denote by $\mathscr{L} = \{\mathscr{L}(T_1, \dots, T_{n-k} | H_t^{t_1,\dots,t_k}), t > 0\}$ the family of survival probability functions of the residual lifetimes of $\mathbf{T}$ conditional on the history $H_t^{t_1,\dots,t_k}$. Finally, we denote by $\mathscr{K} = \{K(\cdot|H_t^{t_1,\dots,t_k}), t > 0\}$ the family of copulas associated with the survival probability functions of $\mathscr{L}$. Following the approach of [9], it is sometimes convenient to express the properties of $\mathscr{K}$ in terms of an equivalent stochastic model that has a simplified form, as expressed in the following result.

Proposition 2.1. *Let $\mathbf{U}$ be a random vector on a suitable probability space, with distribution function equal to the copula C and k–th order statistics denoted by $U_{k:m}$. Let $C(\cdot|\widetilde{H}_z^{z_1,\dots,z_k})$ be the copula associated with $\mathscr{L}(\mathbf{U}|\widetilde{H}_z^{z_1,\dots,z_k})$, where, for $i = 0, 1, \dots, k, z_i = \overline{G}(t_i)$ and $1 > z_1 > \dots > z_k$,*

$$\widetilde{H}_z^{z_1,\dots,z_k} = \{U_{k:n} = z\}$$

for some $z \in (0, z_k)$. Denote $\widetilde{\mathscr{C}} = (C(\cdot|\widetilde{H}_z^{z_1,\dots,z_k}), z \in (0,1])$. Then the mapping

$$\mathscr{K} \to \widetilde{\mathscr{C}}, \quad K(\cdot|H_t^{t_1,\dots,t_k}) \mapsto C(\cdot|\widetilde{H}_z^{z_1,\dots,z_k}),$$

where $z = \overline{G}(t)$, is bijective.

Proof. The proof is a direct consequence of Sklar's Theorem (see, e.g., [34,7]) and the results of [9]. □

In the following, if no confusion may arise, we adopt the convention $H_t^{t_1,\dots,t_k} = H_t^k$ and $\widetilde{H}_z^{z_1,\dots,z_k} = \widetilde{H}_z^k$. Notice that $K = K(\cdot|H_{t=0}^0) = C(\cdot|\widetilde{H}_{z=1}^0) = C$.

The equivalence of the two models of Proposition 2.1 will be extensively used in the remaining part of the work in order to prove our results.

3. Dependence structure of residual lifetimes

In this section we give the expression of the survival copula of residual lifetimes both when one component fails and after this failure. In order to provide meaningful expressions, some assumptions are needed.

All the survival copulas we are considering are supposed to be exchangeable (since they are associated to exchangeable lifetimes) and absolutely continuous. Moreover, all their mixed partial derivatives, when mentioned, are supposed to be continuous and with invertible one-dimensional sections.

We denote by $\frac{\partial C}{\partial u_i}(\mathbf{u})$, or simply $\partial_i C(\mathbf{u})$, the partial derivative of C with respect to the i-th variable, computed at the point $\mathbf{u} \in [0,1]^n$. Analogous notations are used for the mixed partial derivatives.