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Symmetries and solutions of hyperbolic mean curvature flow with a constant forcing term ☆

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ABSTRACT

In this paper we investigate the group-invariant solutions of the hyperbolic mean curvature flow with a constant forcing term by the application of Lie group in differential equations. Based on the associated vector of the obtained symmetry, we construct the group-invariant optimal system of the hyperbolic Monge–Ampère equation.

1. Introduction

The hyperbolic mean curvature flow, i.e., the hyperbolic version of mean curvature flow has been introduced by some authors (see Chou and Wo [2], He, Kong and Liu [3], Kong [4], Kong, Liu and Wang [5], Lefloch and Smoczyk [6], Notz [7]). In fact, Yau in [10] has suggested the following equation related to a vibrating membrane or the motion of a surface

$$\frac{\partial^2 X}{\partial t^2} = H\vec{n}, \tag{1.1}$$

where H is the mean curvature and $\vec{n}$ is the unit inner normal vector of the surface and pointed out that very little about the global time behavior of the hypersurfaces.

Recently, Wang [9] studied the closed convex evolving plane curves under the hyperbolic mean curvature flow with a constant forcing term.

$$\frac{\partial^2 F}{\partial t^2}(u,t) = (k(u,t) - h(u))\vec{N}(u,t) - \nabla\rho \triangleq \left(\frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t}\right)\vec{T}, \quad \forall\ (u,t) \in S^1 \times [0,T), \tag{1.2}$$

where k is the mean curvature, $h(u)$ is a forcing term, $\vec{N}$ and $\vec{T}$ are the unit inner normal at $F(u,t)$ respectively, and s the arclength parameter.

This system is an initial value problem for a system of partial differential equations for F, which can be completely reduced to an initial value problem for a single partial differential equation for its support function. The latter equation is a hyperbolic Monge–Ampère equation.

As we know, symmetry group techniques provide one powerful method for obtaining solutions to partial differential equations (see [1,8]). The methods of finding group-invariant solutions and generalizing the well-known techniques for finding similarity solutions, provide a systematic computational method for determining large classes of special solutions. These

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group-invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations. The more symmetrical the solution is, the easier it is to construct. However, there is almost always an infinite number of the subgroups, and we need an optimal system to classify all possible group-invariant solutions to the system (see [8]).

In this paper, based on the theory of group invariant solutions, we construct the group-invariant optimal system of the hyperbolic Monge–Ampère equation, from which the interesting exact solutions are obtained.

The paper is organized as follows. In Section 2, a hyperbolic Monge–Ampère equation will be derived. In Section 3, we will study the symmetries and exact solutions of the hyperbolic Monge–Ampère equation. Finally, we depict the shape of S under different group-invariant solutions.

2. Hyperbolic Monge–Ampère equation

In the study of the motion of convex curves, it is useful to express the flow in the terms of the support function rather than the graph. Consider (1.2) with the forcing term $h(u) \equiv d$, where d is a constant, by the support function of the curves, we get the following equation (see [9])

$$SS_{\tau\tau} - dS_{\theta\theta} + S_{\tau\tau}S_{\theta\theta} - S_{\theta\tau}^2 + 1 - dS = 0, \quad \forall\ (\theta,\tau) \in \mathbb{R}^2. \tag{2.1}$$

Then, for an unknown function $z = z(\theta,\tau)$ defined for $(\theta,\tau) \in \mathbb{R}^2$, the corresponding Monge–Ampère equation reads

$$A + Bz_{\tau\tau} + Cz_{\tau\theta} + Dz_{\theta\theta} + E(z_{\tau\tau}z_{\theta\theta} - z_{\theta\tau}^2) = 0, \tag{2.2}$$

the coefficients $A,\ B,\ C,\ D$ and E depend on $\tau,\ \theta,\ z,\ z_\tau,\ z_\theta$. We say that the Eq. (2.2) is τ- hyperbolic for z, if

$$\triangle^2(\tau,\theta,z,z_\tau,z_\theta) \triangleq C^2 - 4BD + 4A > 0.$$

It is easy to see that the Eq. (2.1) is a hyperbolic Monge–Amère equation, in which

$$A = 1 - dS, \quad B = S, \quad C = 0, \quad D = -d, \quad E = 1.$$

In fact,

$$\triangle^2(\tau,\theta,S,S_\tau,S_\theta) = C^2 - 4BD + 4A = 0^2 - 4S \times (-d) + 4 \times (1 - dS) = 4 > 0.$$

3. Main results

We consider the one-parameter group of infinitesimal transformations in $\theta,\ \tau,\ S$ given by

$$\begin{aligned} \theta^* &= \theta + \varepsilon\xi(\theta,\tau,S) + o(\varepsilon^2), \\ \tau^* &= \tau + \varepsilon\eta(\theta,\tau,S) + o(\varepsilon^2), \\ S^* &= S + \varepsilon\phi(\theta,\tau,S) + o(\varepsilon^2), \end{aligned} \tag{3.1}$$

where ε is a group parameter.

It is required that Eq. (2.1) be invariant under transformation (3.1), and this yields a system of over determined, linear equations for the infinitesimals $\xi,\ \eta$ and ϕ. Solving these equations, one can get

$$\xi = c_1, \quad \eta = c_2, \quad \phi = c_3\sin(\theta) + c_4\cos(\theta),$$

where $c_i (i = 1,2,3,4)$ are arbitrary constants.

And the associated vector fields for the one-parameter Lie group of infinitesimal transformations are

$$\begin{cases} v_1 = \partial_\theta, \\ v_2 = \partial_\tau, \\ v_3 = \sin(\theta)\partial_S, \\ v_4 = \cos(\theta)\partial_S. \end{cases} \tag{3.2}$$

So the following transformations given by $\exp(\varepsilon v_i),\ i = 1,2,3,4$ of variables $(\theta,\ \tau,\ S)$ leave the solutions of Eq. (2.1) invariant,

$$\begin{aligned} G_1 &= exp(\varepsilon v_1) : (\theta,\tau,S) \to (\theta+\varepsilon,\tau,S), \\ G_2 &= exp(\varepsilon v_2) : (\theta,\tau,S) \to (\theta,\tau+\varepsilon,S), \\ G_3 &= exp(\varepsilon v_3) : (\theta,\tau,S) \to (\theta,\tau,S+\varepsilon\sin(\theta)), \\ G_4 &= exp(\varepsilon v_4) : (\theta,\tau,S) \to (\theta,\tau,S+\varepsilon\cos(\theta)). \end{aligned}$$