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Periodic and subharmonic solutions for fourth-order nonlinear difference equations ☆

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ABSTRACT

By using the critical point theory, some new criteria are obtained for the existence and multiplicity of periodic and subharmonic solutions to fourth-order nonlinear difference equations. The main approach used in our paper is a variational technique and the Linking Theorem. Our results generalize and improve the existing ones.

1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For $a,\ b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \leqslant b$. $*$ denotes the transpose of a vector.

In this paper, we consider the following forward and backward difference equation

$$\Delta^2\left(r_{n-2}\Delta^2 u_{n-2}\right) = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, r_n is real valued for each $n \in \mathbf{Z}$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, r_n and $f(n, v_1, v_2, v_3)$ are T-periodic in n for a given positive integer T.

We may think of (1.1) as a discrete analogue of the following fourth-order functional differential equation

$$[r(t)u''(t)]'' = f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbf{R}. \tag{1.2}$$

Eq. (1.2) includes the following equation

$$u^{(4)}(t) = f(t,\ u(t)), \quad t \in \mathbf{R}, \tag{1.3}$$

which is used to model deformations of elastic beams [7,26]. Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [28].

Difference equations occur widely in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For the general

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background of difference equations, one can refer to monographs [1,3,18]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [12,18,21,36] and results on oscillation and other topics, see [1–4,15–17,19,20,32–35].

The motivation of this paper is as follows. The widely used tools for the existence of periodic solutions of difference equations are the various fixed point theorems in cones. See, for example, [1,3,18] and references therein. It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [5,7,10,11,22,26,30]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [15–17] and Shi et al. [27] have successfully proved the existence of periodic solutions of second-order nonlinear difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations, and in particular, fourth-order equations, has received considerably less attention (see, for example, [1,8,9,13,18,24,25,29,31] and the references contained therein). Yan and Liu [31] in 1997 and Thandapani, Arockiasamy [29] in 2001 studied the following fourth-order difference equation of form,

$$\Delta^2\left(r_n\Delta^2 u_n\right)+f(n,u_n)=0,\quad n\in\mathbf{Z},\tag{1.4}$$

the authors obtain criteria for the oscillation and nonoscillation of solutions for Eq. (1.4). In 2005, Cai et al. [6] have obtained some criteria for the existence of periodic solutions of the fourth-order difference equation

$$\Delta^2\left(r_{n-2}\Delta^2 u_{n-2}\right)+f(n,u_n)=0,\quad n\in\mathbf{Z}.\tag{1.5}$$

In 1995, Peterson and Ridenhour considered the disconjugacy of Eq. (1.5) when $r_n\equiv 1$ and $f(n,u_n)=q_nu_n$ (see [24]). However, to the best of our knowledge, the results on periodic solutions of fourth-order nonlinear difference equations are very scarce in the literature. We found that [6] is the only paper which deals with the problem of periodic solutions to fourth-order difference equation (1.5). Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. The main purpose of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to fourth-order nonlinear difference equations. The proof is based on the Linking Theorem in combination with variational technique. In particular, our results not only generalize the results in the literature [6], but also improve them. In fact, one can see the following Remarks 1.2 and 1.4 for details.

Let

$$\underline{r}=\min_{n\in\mathbf{Z}(1,T)}\{r_n\},\quad \bar{r}=\max_{n\in\mathbf{Z}(1,T)}\{r_n\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied*:

$(r) r_n>0,\ \forall n\in\mathbf{Z}$;

(F_1) *there exists a functional* $F(n,v_1,v_2)\in C^1(\mathbf{Z}\times\mathbf{R}^2,\mathbf{R})$ *with* $F(n,v_1,v_2)\geqslant 0$ *and it satisfies*

$$F(n+T,v_1,v_2)=F(n,v_1,v_2),$$
$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(n,v_1,v_2)}{\partial v_2}=f(n,v_1,v_2,v_3);$$

(F_2) *there exist constants* $\delta_1>0,\ \alpha\in(0,\frac{1}{4}\underline{r}\lambda_{\min}^2)$ *such that*

$$F(n,v_1,v_2)\leqslant\alpha\left(v_1^2+v_2^2\right),\quad \text{for } n\in\mathbf{Z}\quad\text{and}\quad v_1^2+v_2^2\leqslant\delta_1^2;$$

(F_3) *there exist constants* $\rho_1>0,\ \zeta>0,\ \beta\in\left(\frac{1}{4}\bar{r}\lambda_{\max}^2,+\infty\right)$ *such that*

$$F(n,v_1,v_2)\geqslant\beta\left(v_1^2+v_2^2\right)-\zeta,\quad \text{for } n\in\mathbf{Z}\quad\text{and } v_1^2+v_2^2\geqslant\rho_1^2,$$

where $\lambda_{\min},\ \lambda_{\max}$ *are constants which can be referred to* (2.7).

Then for any given positive integer $m>0$, (1.1) *has at least three mT-periodic solutions.*

Remark 1.1. By (F_3) it is easy to see that there exists a constant $\zeta'>0$ such that

$$(F_3')\,F(n,v_1,v_2)\geqslant\beta\left(v_1^2+v_2^2\right)-\zeta',\quad\forall(n,v_1,v_2)\in\mathbf{Z}\times\mathbf{R}^2.$$

As a matter of fact, let $\zeta_1=\max\left\{\left|F(n,v_1,v_2)-\beta\left(v_1^2+v_2^2\right)+\zeta\right|:n\in\mathbf{Z},\ v_1^2+v_2^2\leqslant\rho_1^2\right\},\ \zeta'=\zeta+\zeta_1$, we can easily get the desired result.

Corollary 1.1. *Assume that* (r) *and* (F_1)–(F_3) *are satisfied. Then for any given positive integer* $m>0$, (1.1) *has at least two nontrivial mT-periodic solutions.*