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Approximation solvability of a class of *A*-monotone implicit variational inclusion problems in semi-inner product spaces

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ABSTRACT

This paper deals with the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces. We construct an iterative algorithm for approximating the solution for the class of implicit variational inclusions problems involving *A*-monotone and *H*-monotone operators by using the generalized resolvent operator technique.

1. Introduction and preliminaries

Variational inequalities have been the subject of considerable research and have profound contributions in a large variety of problems arising in mechanics, physics, optimization and control theory, economics and transportation equilibrium problems, and engineering sciences. In [16], Stampacchia introduced the classical variational inequality problem. Because of its wide applications, the classical variational inequality problem has been generalized in different directions. Variational inclusion problem is one of the generalizations of interest and importance. To obtain an efficient and implementable algorithm for solving such problems is always challenging. The methods used for finding solutions of such problems include projection method and its variants, linear approximation methods, method of steepest descent, Newton's methods, and the methods based on auxiliary principle techniques.

The method based on the resolvent operator technique is a generalization of projection method and has been widely used to solve variational inclusion problems (see [1,3–5,8,11]). It is known that the monotonicity of the underlying operator plays a prominent role in solving these problems. Fang and Huang [5] introduced and studied a new class of variational inclusions involving *H*-monotone operators in a Hilbert space. They have obtained a new algorithm for solving the associated class of variational inclusions using resolvent operator technique. Peng and Zhu [13] introduced and studied a new system of generalized mixed quasi-variational inclusions with (H,η)-monotone operators in a Hilbert space. They proved the existence of solutions for the system of generalized mixed quasi-variational inclusions. They also constructed a new iterative algorithm approximating its solution. A considerable research in approximation solvability and *A*-monotone operators have

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been carried out by He et al. [7], Lan et al. [10], Verma [17,19], etc. Verma [17] has considered a class of nonlinear variational inclusions involving A-monotone mappings in a Hilbert space and discussed its solvability using the generalized resolvent operator technique.

In this paper, we prove the existence of solutions for a class of nonlinear implicit variational inclusion problems involving A-monotone operators in a semi-inner product space. By using the generalized resolvent operator technique we construct an iterative algorithm to approximate the solution. As a by product, we extend the work of Verma [17] to a class of semi-inner product spaces. In order to obtain our results we need the concept of monotonicity and generalized monotonicity of an operator in the sense of semi-inner product.

We shall need the following definitions which we quote from the literature:

Definition 1.1 (Lumer [12]). Let X be a vector space over the field F of real or complex numbers. A functional $[.,.] : X \times X \to F$ is called a semi-inner product if it satisfies the following:

1. $[x+y,z] = [x,z] + [y,z], \ \forall x,y,z \in X$;
2. $[\lambda x,y] = \lambda[x,y], \ \forall \lambda \in F$ and $x,y \in X$;
3. $[x,x] > 0$, for $x \neq 0$;
4. $|[x,y]|^2 \leqslant [x,x][y,y]$.

The pair $(X, [.,.])$ is called a semi-inner product space.

We observe that $\|x\| = [x,x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [6] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to define a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x,y] = 0$ if and only if y is orthogonal to x, that is if and only if $\|y\| \leqslant \|y + \lambda x\|$, for all scalars λ.
(ii) Generalized Riesz representation theorem:- If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x,y]$, for all $x \in X$.
(iii) The semi-inner product is continuous, that is for each $x,y \in X$, we have $Re[y, x+\lambda y] \to Re[y,x]$ as $\lambda \to 0$.

The sequence space $l^p, \ p > 1$ and the function space $L^p, \ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 1.1. The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x,y] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \quad x,y \in l^p.$$

Example 1.2 (Giles [6]). The real Banach space $L^p(X,\mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f,g] = \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn(g(x)) d\mu, \quad f,g \in L^p.$$

Definition 1.2 (Xu [21]). Let X be a real Banach space. The modulus of smoothness of X is defined as

$$\rho_X(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \ \|x\| = 1, \|y\| = t, \ t > 0\right\}.$$

X is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$.
X is said to be p-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leqslant ct^p, \ p > 1$.
X is said to be 2-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leqslant ct^2$.

Lemma 1.1 (Xu [21]). *Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) *X is 2-uniformly smooth.*
(ii) *There is a constant $c > 0$ such that for every $x,y \in X$, the following inequality holds*