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Rothe's method for solving some fractional integral diffusion equation

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ABSTRACT

In this paper, we apply the Rothe's method to a fractional integral diffusion equation and establish the existence and uniqueness of a strong solution. As an application, we include an example to illustrate the main result.

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1. Introduction

In this paper we apply the Rothe's method to the following fractional integral diffusion equation in a Banach space X

$$\frac{\partial u(t)}{\partial t} + Au(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds + f(t), \quad t \in (0,T], \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $0 < \alpha < 1$, $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, f is a given map from $[0,T]$ into X, $u_0 \in D(A) \subset A$, the domain of A.

The problem considered in this paper is a particular case of the fractional integral diffusion problem

$$D^\beta u(t) + Au(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds + f(t), \quad u(0) = u_0,$$

where $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$. If we take $\beta = 1$ and $0 < \alpha < 1$, then above problem reduces to the problem (1) and (2).

In 1930, E. Rothe [8] has introduced a method to solve the following scalar parabolic initial boundary value problem of second order

$$R(t,x)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = S(t,x,u), \quad 0 < x < 1, \ t > 0,$$

$$u(0,x) = u_0(x),$$

$$u(t,0) = u(t,1) = 0, \quad t \geqslant 0,$$

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where R and S are sufficiently smooth functions of t and x in $[0,T]\times(0,1)$ satisfying certain additional conditions. Here T means an arbitrary finite positive number. His method consist in dividing $[0,T]$ into n number of subintervals $[t^n_{j-1},t^n_j],\ t^n_j=jh,\ j=1,2,\ldots,n$ with $t^n_0=0$ of equal lengths $h(h=\frac{T}{n})$ and replacing the partial derivative $\frac{\partial u}{\partial t}$ of the unknown function u by the difference quotients $\frac{u^n_j-u^n_{j-1}}{h}$. After defining a sequence of polygonal functions

$$U^n(x,t)=u^n_{j-1}(x)+\frac{1}{h}(t-t^n_{j-1})(u^n_j(x)-u^n_{j-1}(x)),\quad t\in[t^n_{j-1},t^n_j].$$

Rothe has proved the convergence of the sequence $\{U^n\}$ to the unique solution of the problem as $n\to\infty$ using some a priori estimates on $\{U^n\}$. The problem treated by Rothe is a simple one but the method introduced by him turns out to be a very powerful theoretical tools for proving the existence and uniqueness of solutions of linear as well as nonlinear parabolic and hyperbolic problems of higher orders. This method is known as "Rothe's method". It is also known as the method of semidiscretization or the method of lines. After that many authors have applied this method to various classical types of initial boundary value problem; for instance [9–12,14–18] and references therein.

Dubey [4], has established the existence and uniqueness of a strong solution for the following nonlinear nonlocal functional differential equation in a Banach X, using the method of semidiscretization:

$$u'(t)+Au(t)=f(t,u(t),u_t),\quad t\in(0,T],$$
$$h(u_0)=\phi\quad\text{on}\quad[-\tau,0],$$

where $0<T<\infty$, $\phi\in C_0:=C([-\tau,0];X),\ \tau>0$, the nonlinear operator A is singlevalued and m-accretive defined from the domain $D(A)\subset X$ into X, the nonlinear map f is defined from $[0,T]\times X\times C_0:=C([-\tau,0];X)$ into X, the map h is defined from C_0 into C_0. For $u\in C_T:=C([-\tau,T];X)$, function $u_t\in C_0$ is given by $u_t(s)=u(t+s)$ for $s\in[-\tau,0]$. Here $C_t:=C([-\tau,t];X)$ for $t\in[0,T]$ is the Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the supremum norm

$$\|\phi\|_t=\sup_{-\tau\leqslant\eta\leqslant t}\|\phi(\eta)\|,\quad \phi\in C_t.$$

For the application of Rothe's method to delay differential equation, delayed cooperation diffusion system, integrodifferential equation, parabolic and hyperbolic problems, we refer the readers to [1–3,5,9–22].

By literature, it is clear that Rothe's method or the method of semidiscretization is applicable in many physical, mathematical, biological problems modeled by partial differential equations.

In the present paper our aim is to apply the Rothe's method to a fractional integral diffusion problem and to establish the existence and uniqueness of a strong solution. This work is motivated by the work of Lin and Xu [23]. In which authors used method based on time discretization to the following time fractional diffusion problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}-\frac{\partial^2 u(x,t)}{\partial x^2}=f(x,t),\quad x\in\wedge,\quad 0<t\leqslant T.$$

Subject to the following initial and boundary conditions

$$u(x,0)=g(x),\quad x\in\wedge,$$

$$u(0,t)=u(L,t)=0,\quad 0\leqslant t\leqslant T,$$

where $0<\alpha<1$ is the order of the time fractional derivative. $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is defined as Caputo fractional derivative of order α, given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}.$$

In [24], authors develop the Crank–Nicolson finite difference method to solve the following linear time-fractional diffusion equation with Dirichlet boundary conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2},$$

$$u(x,0)=f(x),$$

$$u(0,t)=u(1,t)=0,$$

where $0<x<1,\ 0\leqslant t\leqslant T$ and the parameter $0<\alpha<1$ refers to the fractional order of the time derivative. For the time discretization in fractional differential equations, we refer the readers to [25].

The plan of the rest paper is as follows. In Section 2, we state some preliminaries and the main result. In Section 3, we state and prove all the lemmas that are required to prove the main result and at the end of this section, we prove the main result. In the last section, as an application, we include an example to illustrate the main result.

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