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Dichotomy of a perturbed Lyness difference equation

Guifeng Deng [a], Xianyi Li [b,*]

[a] School of Mathematics and Information, Shanghai Lixin University of Commerce, Shanghai 201620, China
[b] School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu Province 225002, China

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ABSTRACT

We investigate in this paper the perturbed Lyness difference equation $bx_{n+2}x_n = \alpha + \beta x_{n+1} + \gamma x_n^2$, $n = 0, 1, 2, \ldots$, where α, β, b are arbitrary positive real numbers and $\gamma \in [0, \infty)$ and the initial values x_1, $x_0 > 0$, which is a generalization of the Lyness difference equation $x_{n+2}x_n = a + x_{n+1}$ extensively studied. It is known that for the Lyness difference equation, i.e., the perturbed Lyness difference equation with $\gamma = 0$, all its solutions are periodic or strictly oscillatory. However, one here finds that this perturbed Lyness difference equation possesses the following dichotomy: for $0 < \gamma < b$, all of its solutions are globally asymptotically stable; for $\gamma \geqslant b$, all the sequences generated by it converge to $+\infty$. We hence find that there exists the essential difference for the properties of solutions between the unperturbed Lyness difference equation and the perturbed Lyness difference equation.

1. Introduction

In recent years the problem of stability of difference equations has been one of the important research areas in dynamical system theory. For example, see [1–9] and the references cited therein. There are different kinds of means and ways for investigating the stability of difference equations. Among them, Lyapunov function is an important one [8,9]. In addition, the invariant curves of difference equations often play a critical role in studying the stability behavior and oscillatory character of solutions of difference equations. Refer to [8–11]. Especially, Kocic et al. [12] have applied KAM theory to prove the stability of the solutions of Lyness equation.

After Lyness [13] found in the process of studying the problem of number theory the Lyness equation

$$x_{n+2}x_n = x_{n+1} + a, \quad a > 0, \quad x_1, x_0 > 0, \tag{1.1}$$

to be periodic with period five for $a = 1$, various generalized Lyness difference equations, including Lyness difference equation with constant coefficients and Lyness difference equation with variable coefficients, have been extensively studied. For example, see [10,11,14,15] and the references cited therein.

The second author of this paper with his co-author in [14] obtained the sufficient and necessary condition for period-five solutions of generalized Lyness difference equation

$$x_{n+2} = \frac{ax_{n+1} + b}{(cx_{n+1} + d)x_n},$$

* Corresponding author.
E-mail address: mathxyli@yzu.edu.cn (X. Li).

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where $a, b, c, d \geqslant 0$ with $a + b > 0,\ c + d > 0$ and the initial values $x_1,\ x_0 > 0$, and the sufficient condition for its oscillation. In [15], the second author of this paper found the convergent solutions of Lyness difference equation with variable coefficients

$$x_{n+2} = \frac{x_{n+1} + a_n}{x_n}, \quad x_1, x_0 > 0,$$

where $\{a_n\}$ is a monotonic decreasing positive sequence. This not only denies a conjecture provided by G.Ladas but also demonstrate the essential difference between Lyness difference equation with constant coefficients and Lyness difference equation with variable coefficients because there are no convergent nontrivial solutions for Lyness difference equation with constant coefficients.

In this paper, we will study a specific perturbation of Lyness Eq. (1.1), namely, the following perturbed Lyness difference equation

$$x_{n+2} = \frac{\alpha + \beta x_{n+1} + \gamma x_n^2}{b x_n}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $\alpha, \beta, b \in (0, +\infty),\ \gamma \in [0, +\infty)$ and the initial values $x_0, x_1 \in (0, +\infty)$.

Notice that the quadratic term x_n^2 is indeed a small perturbation for small positive number γ. From here one will observe the preservation of global stable behavior of solutions generated by a difference equation under small perturbation.

Our main results in this paper are as follows.

Theorem 1.1. *The following statements are true.*

(1) *If $\gamma = 0$, then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Eq. (1.2) is periodic or strictly oscillatory. Moreover, the equilibrium point l_0 is locally stable, where*

$$l_0 = \frac{\beta + \sqrt{\beta^2 + 4\alpha b}}{2b}.$$

i.e., for every $\epsilon > 0$ there is a $\delta > 0$ so that for any positive initial values x_0 and x_1 with $|x_i - l_0| < \delta$ for $i = 0, 1$, one has $|x_n - l_0| < \epsilon$ for all $n \geqslant 0$.

(2) *If $0 < \gamma < b$, then the positive equilibrium point l_γ of Eq. (1.2) is globally asymptotically stable, where*

$$l_\gamma = \frac{\beta + \sqrt{\beta^2 + 4\alpha(b - \gamma)}}{2(b - \gamma)}.$$

(3) *If $\gamma \geqslant b$, then any one solution $\{x_n\}_{n=0}^{\infty}$ of Eq. (1.2) converges to $+\infty$.*

It follows from the above theorem that the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent to l_γ for $0 < \gamma < b$, and whereas diverges to $+\infty$ for $\gamma \geqslant b$. So the qualitative nature of solutions of Eq. (1.2) changes as γ increases. Note that the limit of sequence $\{x_n\}_{n=0}^{\infty}$ as n tends to $+\infty$ is $\lim_{n\to+\infty} x_n = l_\gamma$ when $0 < \gamma \ll b$. So the behavior of the solutions of Eq. (1.2) is completely different from the unperturbed case, that is, Eq. (1.1). Thus, the values 0 and b are two bifurcation points for Eq. (1.2) containing the parameter γ.

Remark 1.2.

1. The (1) and (2) in the above Theorem 1.1 actually demonstrate the preservation of global stable behavior of solutions generated by Eq. (1.1) under small perturbation.
2. The (2) and (3) in the above Theorem 1.1 show the dichotomy of Eq. (1.2).
3. Without loss of generality, the b in Eq. (1.2) may be assumed to be 1.

2. First integral and Lyapunov function

In this section we investigate the stability character of solutions of Eq. (1.2). It is easy to see that for $0 \leqslant \gamma < b$ Eq. (1.2) has a unique equilibrium point l_γ given by

$$l_\gamma = \frac{\beta + \sqrt{\beta^2 + 4\alpha(b - \gamma)}}{2(b - \gamma)}.$$

Eq. (1.2) has an associated nonlinear map

$$F(x, y) = (y, \frac{\alpha + \beta y + \gamma x^2}{b x}). \tag{2.1}$$