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Two-sided approximation for some Newton's type methods

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ABSTRACT

We suggest and analyze a combination of a damped Newton's method and a simplified version of Newton's one. We show that the proposed iterations give two-sided approximations of the solution which can be efficiently used as posterior estimations. Some numerical examples illustrate the efficiency and performance of the method proposed.

1. Introduction

In the last decade, new iterative methods containing parameters for a numerical solving of nonlinear equations have been developed by many authors. The role of these parameters play, for example, a damped parameter in Newton type methods [1–6], interpolation nodes in inverse polynomial interpolation methods [7,8]. They can be controlled not only by the convergence order, but also by the convergence behavior. One of the advantages of such methods is that they give two-sided approximations of the solutions which allow one to control the error at each iteration step [8,3,6]. In this paper we will consider a combination of the damped Newton's method and the simplified Newton method.

The paper is organized as follows. In Section 2 we formulate new iteration schemes for solving nonlinear equations. In Section 3 we show that the proposed iterations give two-sided approximation of the solution. In Section 4 we prove that the convergence order of these iterations is at least 2. Depending on the suitable choices of parameters, the convergence order may be increased from 2 to 4. Some numerical examples illustrating the theoretical results are given in Section 5.

2. Statement of the problem

Let $a, b \in \mathbb{R},\ a < b,\ f : [a, b] \to \mathbb{R}$ and consider the following nonlinear equation

$$f(x) = 0. \tag{1}$$

Assume that $f(x) \in C^3[a, b],\ f'(x) \neq 0,\ x \in [a, b]$ and Eq. (1) has a unique root $x^* \in [a, b]$. For a numerical solution of Eq. (1) we propose the following iterations

$$x_{2n+1} = x_{2n} - \tau_n \frac{f(x_{2n})}{f'(x_{2n})}, \quad n = 0, 1, \ldots, \tag{2a}$$

$$x_{2n+2} = x_{2n+1} - \omega_n f(x_{2n+1}). \tag{2b}$$

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Here $\tau_n > 0$ and ω_n are the iteration parameters to be determined properly. It should be mentioned that the first iteration (2a) is a continuous analogy of Newton's method (or damped Newton's method), while the second one (2b) is a simple iteration. In [6] it is shown that the iterations (2a) and (2b) with

$$\omega_n = \frac{1}{f'(x_{2n+1})} \tag{3}$$

have a two-sided approximation behavior, and it proved the convergence rate of these iterations is 4 when $\tau_n \to 1$ as $n \to \infty$.

On the other hand, the iterations (2a) and (2b) can be considered as simple iterations

$$x_{2n+1} = p(x_{2n}), \quad x_{2n+2} = q(x_{2n+1}), \quad n = 0, 1, \ldots, \tag{4}$$

for two equations

$$x - p(x) = 0, \quad x - q(x) = 0, \tag{5}$$

which are equivalent to the above Eq. (1) and with functions

$$p(x) = x - \tau\frac{f(x)}{f'(x)}, \quad q(x) = x - \omega f(x). \tag{6}$$

3. The convergence of the proposed iterations

Suppose that [6]

$$\left|\frac{f''(x)}{(f'(x))^2}f(x)\right| \leqslant M_2\left|\frac{f(x)}{(f'(x))^2}\right| \leqslant a(x) < \frac{4}{9}, \quad x \in [a,b], \tag{7}$$

where $M_2 = \max_{x\in[a,b]}|f''(x)|$. Then it is easy to show that the function $p(x)$ satisfies

$$0 < p'(x_{2n}) < 1, \quad n = 0, 1, \ldots, \tag{8}$$

under condition

$$\tau_n \in \left(0, \frac{1}{1 - a_{2n}}\right), \quad a_{2n} = M_2\left|\frac{f(x_{2n})}{(f'(x_{2n}))^2}\right|. \tag{9}$$

A sufficient condition for $q(x)$ to be decreasing is

$$\omega_n f'(x_{2n+1}) > 1, \quad n = 0, 1, \ldots \tag{10}$$

It should be noted that the conditions (8) and (10) were used first in [7,8] for bilateral approximations of Aitken–Steffensen–Hermite type methods.

Using Taylor expansion of $f(x_{2n+2})$ at point x_{2n+1}, and (2b), we obtain

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} = 1 - \omega_n f'(x_{2n+1}) + \frac{f''(\xi_{2n})}{2}f(x_{2n+1})\omega_n^2, \tag{11}$$

where $\xi_{2n} = \theta x_{2n+2} + (1-\theta)x_{2n+1}, \ \theta \in (0,1)$.

Lemma 1. *Suppose that*

$$f''(\xi_{2n})f(x_{2n+1}) < 0, \quad n = 0, 1 \tag{12}$$

and the inequality (10) holds. Then

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 0, \quad n = 0, 1, \ldots \tag{13}$$

Proof. If to take (12) and (10) into account, then from formula (11) we get

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 1 - \omega_n f'(x_{2n+1}) < 0. \tag{14}$$

The Lemma is proved. □

Analogously, using Taylor expansion of $f(x_{2n+1})$ at point x_{2n}, and (2a) we obtain

$$\frac{f(x_{2n+1})}{f(x_{2n})} = 1 - \tau_n + \frac{f''(\eta_{2n})}{2}\frac{f(x_{2n})}{(f'(x_{2n}))^2}\tau_n^2, \quad n = 0, 1, \ldots, \tag{15}$$