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# Asymptotics for Laguerre–Sobolev type orthogonal polynomials modified within their oscillatory regime

Edmundo J. Huertas [a,*,1], Francisco Marcellán [b,2], M. Francisca Pérez-Valero [b,3], Yamilet Quintana [c,4]

[a] CMUC, Departamento de Matemática, FCTUC, Universidade de Coimbra, Largo D. Dinis, Apdo. 3008, 3001-544 Coimbra, Portugal
[b] Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Madrid, Spain
[c] Departamento de Matemáticas Puras y Aplicadas, Universidad Simón Bolívar, Postal Code 89000, Caracas 1080 A, Venezuela

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ABSTRACT

In this paper we consider sequences of polynomials orthogonal with respect to the discrete Sobolev inner product

$$\langle f,g\rangle_S = \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + \mathbb{F}(c)A\mathbb{G}(c)^t, \quad \alpha > -1,$$

where $f$ and $g$ are polynomials with real coefficients, $A \in \mathbb{R}^{(2,2)}$ and the vectors $\mathbb{F}(c)$, $\mathbb{G}(c)$ are

$$A = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \quad \mathbb{F}(c) = (f(c), f'(c)) \quad \text{and } \mathbb{G}(c) = (g(c), g'(c)), \text{respectively},$$

with $M, N \in \mathbb{R}_+$ and the mass point $c$ is located inside the oscillatory region for the classical Laguerre polynomials. We focus our attention on the representation of these polynomials in terms of classical Laguerre polynomials and we analyze the behavior of the coefficients of the corresponding five-term recurrence relation when the degree of the polynomials is large enough. Also, the outer relative asymptotics of the Laguerre–Sobolev type with respect to the Laguerre polynomials is analyzed.



## 1. Introduction

The study of asymptotic properties for general orthogonal polynomials is an important challenge in approximation theory and their applications permeate many fields in science and engineering [30,32,40,41]. Although it may seem as an old subject from the point of view of standard orthogonality [5,41], this is not the case neither in the general setting (cf. [16,17,30,

* Corresponding author.
*E-mail address:* ehuertas@mat.uc.pt (E.J. Huertas).

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36–38,40]) nor from the viewpoint of Sobolev orthogonality, where it remains like a partially explored subject [3]. In fact, in the last ten years this topic has attracted the interest of many researchers [4,7–10,12,13,19,22–24,26,33–35].

A Sobolev-type or discrete Sobolev-type inner product on the linear space $\mathbb{P}$ of polynomials with real coefficients is defined by

$$\langle f,g\rangle_S = \int f(x)g(x)d\mu_0(x) + \sum_{k=0}^{d}\mathbb{F}(c_k)A_k\mathbb{G}(c_k)^t, \quad d\in\mathbb{Z}_+, \tag{1}$$

where $\mu_0$ is a nontrivial finite and positive Borel measure supported on the real line, $f,g\in\mathbb{P}$, and for $k=0,\dots,d,\ d\in\mathbb{Z}_+$, the matrices $A_k=(a_{ij}^{(k)})\in\mathbb{R}^{(1+N_k)(1+N_k)}$ are positive semi-definite. We denote by $\mathbb{F}(c_k)$ and $\mathbb{G}(c_k)$ the vectors $\mathbb{F}(c_k)=\big(f(c_k),f'(c_k),\dots,f^{(N_k)}(c_k)\big)$ and $\mathbb{G}(c_k)=\big(g(c_k),g'(c_k),\dots,g^{(N_k)}(c_k)\big)$, respectively, with $c_k\in\mathbb{R},\ N_k\in\mathbb{Z}_+$ and, as usual, $v^t$ denotes the transpose of the vector $v$. This notion was initially introduced in [11] for diagonal matrices $A_k$ in order to study recurrence relations for sequences of polynomials orthogonal with respect to (1).

The study of asymptotic properties of the sequences of orthogonal polynomials with respect to particular cases of the inner product (1) has been done by considering separately the cases 'mass points inside' or 'mass points outside' of *supp* $\mu_0$, respectively, being *supp* $\mu_0$ a bounded interval of $\mathbb{R}$ or, more recently, an unbounded interval of the real line (see, for instance [7–10,12,19,26]). The first results in the literature about asymptotic properties of orthogonal polynomials with respect to a Sobolev-type inner product like (1) appear in [27], where the authors considered $d=0,\ N_0=1,\ a_{11}^{(0)}=a_{12}^{(0)}=a_{21}^{(0)}=0,\ a_{22}^{(0)}=\lambda$, with $\lambda>0$. Therein, such asymptotic properties when there is only one mass point supporting the derivatives either inside or outside $[-1,1]$ and $\mu$ is a measure in the Nevai class $M(0,1)$ are studied.

In [17], using an approach based on the theory of Padé approximants, the authors obtain the outer relative asymptotics for orthogonal polynomials with respect to the Sobolev-type inner product (1) assuming that $\mu_0$ belongs to Nevai class $M(0,1)$ and the mass points $c_k$ belong to $\mathbb{C}\setminus$ *supp* $\mu$. The same problem with the mass points in *supp* $\mu=[-1,\,1]$ was solved in [39], provided that $\mu'(x)>0$ a.e. $x\in[-1,\,1]$ and $A_k$ being diagonal matrices with $a_{ii}^{(k)}$ non-negative constants. The pointwise convergence of the Fourier series associated to such an inner product was studied when $\mu_0$ is the Jacobi measure (see also [20,21]). On the other hand, the asymptotics for orthogonal polynomials with respect to the Sobolev-type inner product (1) with $\mu_0\in M(0,1),\ c_k$ belong to *supp* $\mu\setminus[-1,\,1]$, and $A_k$ are complex diagonal matrices such that $a_{1+N_k,1+N_k}^{(k)}\neq 0$, was solved in [2].

Another results about the asymptotic behavior of orthogonal polynomials associated with diagonal (resp. non-diagonal) Sobolev inner products with respect to measures supported on the complex plane can be found in [1,4,7,28]. On the other hand, results concerning asymptotics for extremal polynomials associated to non-diagonal Sobolev norms may be seen in [29,33–35].

In this paper we deal with sequences of polynomials orthogonal with respect to a particular case of (1). Indeed, $\mu_0$ is the Laguerre classical measure

$$\langle f,g\rangle_S = \int_0^{\infty} f(x)g(x)x^{\alpha}e^{-x}dx + \mathbb{F}(c)A\mathbb{G}(c)^t, \quad \alpha>-1, \tag{2}$$

$f,g\in\mathbb{P}$. The matrix $A$ and the vectors $\mathbb{F}(c),\ \mathbb{G}(c)$ are

$$A=\begin{pmatrix} M & 0\\ 0 & N\end{pmatrix}, \quad \mathbb{F}(c)=(f(c),f'(c)) \quad \text{and } \mathbb{G}(c)=(g(c),g'(c)),\ \text{respectively},$$

$M,N\in\mathbb{R}_+$, and the mass point $c$ is located inside the oscillatory region for the classical Laguerre polynomials, i.e., $c>0$. Following the methodology given in [7–10,19,26] we focus our attention on the representation of these polynomials in terms of the classical Laguerre polynomials. Their asymptotic behavior will be discussed.

More precisely, as it was mentioned above, recent works like [7–10,19,26] have focused the attention on the study of asymptotic properties of the sequences of orthogonal polynomials with respect to specific cases of the inner product (1) with 'mass points outside' of *supp* $\mu_0$, being *supp* $\mu_0$ an unbounded interval of the real line. However, to the best of our knowledge, asymptotic properties of the sequences of orthogonal polynomials associated to (2) are not available in the literature.

The structure of the manuscript is as follows. Section 2 contains the basic background about Laguerre polynomials and some other auxiliary results which will be used throughout the paper. In Section 3 we prove our main result, namely the outer relative asymptotic of the Laguerre–Sobolev type orthogonal polynomials modified into the *positive* real semiaxis. Finally, in Section 4 we deduce the coefficients of the corresponding five-term recurrence relation as well as their asymptotic behavior when the degree of the polynomials is large enough.

Throughout this manuscript, the notation $u_n\sim v_n$ means that the sequence $\{\frac{u_n}{v_n}\}_n$ converges to 1 as $n\to\infty$. Any other standard notation will be properly introduced whenever needed.

## 2. Background and previous results

Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle f,g\rangle_{\alpha} = \int_0^{\infty} f(x)g(x)x^{\alpha}e^{-x}dx, \quad \alpha>-1, \quad f,g\in\mathbb{P}. \tag{3}$$