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Nilpotent singular points and compactons

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ABSTRACT

In this paper, dynamical system theory is applied to several types of fully nonlinear wave equations. These equations can be reduced to planar polynomial differential systems by transformation of variables. We treat these polynomial differential systems by phase space analytical technique. The results of our study demonstrate that there exist close connection between nilpotent singular points and compactons. Moreover, we find some new elliptic function compactons instead of well-known trigonometric function compactons by analyzing nilpotent points. Two new compactons induced by singular elliptic are also obtained.

1. Introduction

It is well known that the study of nonlinear wave equations and their solitary wave solutions are of great importance in many areas of physics. Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytic formulae (typically a sech^2 function or variants thereof) and serve as prototypical solutions that model physical localized waves. In the case of integrable systems, the solitary waves interact cleanly, and are known as solitons. For many examples, localized initial data ultimately breaks up into a finite collection of solitary wave solutions; this fact has been proved analytically for certain integrable equations such as the Korteweg-de Vries equation, and is observed numerically in many others. The appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations, including peakons [1–5], which have a corner at their crest, cuspons [2], having a cusped crest, and, compactons [6–13], which have compact support, has vastly increased the menagerie of solutions appearing in model equations, both integrable and non-integrable. The distinguishing feature of the systems admitting non-analytic solitary wave solutions is that, in contrast to the classical nonlinear wave equations, they all include a nonlinear dispersion term, meaning that the highest order derivatives (characterizing the dispersion relation) do not occur linearly in the system, but are typically multiplied by a function of the dependent variable.

There are two important nonlinearly dispersive equations. One is the well-known Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

which was proposed by Camassa and Holm [1] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa–Holm equation has been found to has peakons, cuspons and composite wave solutions [2]. The other is the $K(m,n)$ equation

$$u_t \pm (u^m)_x + (u^n)_{xxx} = 0, \tag{1.2}$$

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which was discovered by Rosenau and Hyman [6]. In particular, the $K(2,2)$ equation support compacton solutions

$$u(x,t) = \begin{cases} \frac{4c}{3}\cos^2(\frac{x-ct}{4}), & |x-ct| \le 2\pi, \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

The compactons represent traveling solitary wave solutions with compact support. That is, they vanish identically outside a finite range. The compactons are also robust within their range of existence.

In general, Eq. (1.2) does not exhibit the usual energy conservation law. Therefore, Cooper et al. [12] proposed a generalization of the KdV equation based on the first-order Lagrangian

$$L(l,p) = \int \left[\frac{1}{2}\phi_x\phi_t + \frac{(\phi_x)^2}{l(l-1)} - \alpha(\phi_x)^p(\phi_{xx})^2\right]dx, \tag{1.4}$$

which leads to the CSS equation

$$u_t + u^{l-2}u_x - p[u^{p-1}(u_x)^2]_x + 2\alpha[u^p u_x]_{xx} = 0. \tag{1.5}$$

In particular, if $p = 1$, $l = 3$ and $\alpha = \frac{1}{2}$, then the CSS equation has also trigonometric function compactons

$$u(x,t) = \begin{cases} 3c\cos^2(\frac{1}{2\sqrt{3}}(x-ct)), & |x-ct| \leqslant \sqrt{3}\pi, \\ 0, & \text{otherwise}. \end{cases} \tag{1.6}$$

Recently, Mihaila et al. [14,15] studied the numerical stability of single compactons of the $K(m,n)$ equation and the CSS equation and their pairwise interactions by using Padé approximant method

In this paper, we consider the following nonlinear wave equation

$$u_t + (A(u))_x + \mu u_{xxt} = [B(u,u_x)u_x^2 + C(u,u_x)u_{xx}]_x. \tag{1.7}$$

where μ is a constant. Some special cases of (1.1) have appeared in the literature.

(1) When $\mu = 0$, $A(u) = u^2$, $B(u,u_x) = -2$ and $C(u,u_x) = -2u$, then Eq. (1.7) becomes the $K(2,2)$ equation [6]

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \tag{1.8}$$

(2) When $\mu = 0$, $A(u) = u^3$, $B(u,u_x) = -2$ and $C(u,u_x) = -2u$, then Eq. (1.7) becomes the $K(3,2)$ equation [6]

$$u_t + (u^3)_x + (u^2)_{xxx} = 0. \tag{1.9}$$

(3) When $\mu = 0, A(u) = u^3, B(u,u_x) = 1$ and $C(u,u_x) = -u$, then Eq. (1.7) is the sinh-Gordon equation [16]

$$u_t + 3u^2u_x - u_xu_{xx} + uu_{xxx} = 0. \tag{1.10}$$

(4) When $\mu = 1, A(u) = a - \frac{1}{2}u^3, B(u,u_x) = \frac{1}{2}u$ and $C(u,u_x) = \frac{1}{2}(u^2 + u_x^2)$, then Eq. (1.7) is changed to the Olver–Rosenau equation [7,17]

$$m_t = au_x + \frac{1}{2}[(u^2 + u_x^2)m]_x, \quad m = u + u_{xx}. \tag{1.11}$$

The paper is organized as follows. In Section 2, we reduce Eq. (1.7) to planar polynomial differential system by transformation of variables and introduce nilpotent singular points of vector field. In Section 3, we find some new compactons by phase space analysis of nilpotent points. A short conclusion is given in Section 4.

2. Vector fields and nilpotent points

A nonlinear evolution equation is often called quasilinear when the dispersive term of it is linear. All quasilinear equations can easily be treated by the plane polynomial differential dynamical systems theory. For example, the change of variable

$$u(x,t) = \varphi(x-ct) = \varphi(\xi) \tag{2.1}$$

followed by an integration over ξ converts the quasilinear KdV equation

$$u_t + \alpha uu_x + \gamma u_{xxx} = 0 \tag{2.2}$$

with parameters α and γ, to an ordinary nonlinear differential equation

$$\frac{d^2\varphi}{d\xi^2} = \frac{c}{\gamma}\varphi - \frac{\alpha}{2\gamma}\varphi^2. \tag{2.3}$$