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A note on tripled coincidence and tripled common fixed point theorems in partially ordered metric spaces ☆

Stojan Radenović

University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11 120 Beograd, Serbia

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ABSTRACT

In this paper we have used a method of reducing tripled coincidence and tripled fixed point results in (ordered) metric spaces to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. Our results generalize, extend, unify, enrich and complement recently tripled coincidence point theorems established by Berinde and Borcut (2011) [4], Borcut and Berinde (2012) [7] and Borcut (2012) [9]. Also, by using our method several coupled coincidence and coupled common fixed point results in ordered metric spaces can be reduced to the coincidence and common fixed point results with one variable.

1. Introduction and preliminaries

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instance: variational inequalities, optimization, approximation theory, etc.

The notion of a tripled fixed point was introduced and studied by Berinde and Borcut [4]. Fixed point theory as well as and coupled and tripled cases in ordered metric spaces was studied in [1–14].

The fixed point theorems in partially ordered metric spaces play a major role to prove the existence and uniqueness of solutions for some differential and integral equations. One of the most interesting fixed point theorems in ordered metric spaces was investigated by Ran and Reurings [17] applied their result to linear and nonlinear matrix equations. Then many authors obtained several interesting results in ordered metric spaces [10–15].

We start out with listing some notation and preliminaries that we shall need to express our results. In this paper $(X, d, \preceq)$ denotes a partially ordered metric space where $(X, \preceq)$ is a partially ordered set and (X, d) is a metric space.

Definition 1.1 ([3,4]). Let $(X, \preceq)$ be a partially ordered set. The mapping $F : X^3 \to X$ is said to have the mixed monotone property if for any $x, y, z \in X, F(x, y, z)$ is monotone non-decreasing in x and z, and monotone non-increasing in y. An element $(x, y, z) \in X^3$ is called a tripled fixed point of F if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.

Definition 1.2 ([9,12]). Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ and $g : X \to X$ two mappings.

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E-mail address: radens@beotel.rs

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(1) The mapping F is said to have the mixed g-monotone property if for any $x,y,z \in X$, $F(x,y,z)$ is monotone g-non-decreasing in x and z and monotone g-non -increasing in y.
(2) An element $(x,y,z) \in X^3$ is called a tripled coincidence point of F and g if $F(x,y,z) = gx$, $F(y,x,y) = gy$ and $F(z,y,x) = gz$. Moreover, (x,y,z) is called a tripled common fixed point of F and g if $F(x,y,z) = gx = x$, $F(y,x,y) = gy = y$ and $F(z,y,x) = gz = z$.
(3) Mappings F and g are called commutative if for all $x,y,z \in X$ holds $g(F(x,y,z)) = F(gx,gy,gz)$.
(4) Let (X,d) be a metric space. Mappings $F : X^3 \to X$ and $g : X \to X$ are called compatible if

$$\lim_{n\to\infty} d(gF(x_n,y_n,z_n), F(gx_n,gy_n,gz_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n,x_n,y_n), F(gy_n,gx_n,gy_n)) = 0, \text{ and}$$
$$\lim_{n\to\infty} d(gF(z_n,y_n,x_n), F(gz_n,gy_n,gx_n)) = 0,$$

hold whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} gx_n, \quad \lim_{n\to\infty} F(y_n,x_n,y_n) = \lim_{n\to\infty} gy_n \text{ and } \lim_{n\to\infty} F(z_n,y_n,x_n) = \lim_{n\to\infty} gz_n.$$

Definition 1.3. Let $(X,\preceq)$ be an ordered set and d be a partial metric on X. We say that $(X,d,\preceq)$ is regular if it has the following properties:

(i) if for non-decreasing sequence $\{x_n\}$ holds $d(x_n,x) \to 0$, then $x_n \preceq x$ for all n,
(ii) if for non-increasing sequence $\{y_n\}$ holds $d(x_n,x) \to 0$, then $y_n \succeq y$ for all n.
The proof of the following Lemma is immediately. We note that the same idea as in this paper, but in the case of coupled fixed point theorems has been first used in [5].

Lemma 1.4.

(1) *Let $(X,d,\preceq)$ be a partially ordered metric space. If relation $\sqsubseteq$ is defined on X^3 by*

$$Y \sqsubseteq X \Longleftrightarrow x \preceq u \wedge y \succeq v \wedge z \preceq w, \quad Y = (x,y,z), \quad V = (u,v,w) \in X^3,$$

and $D_1, D_2 : X^3 \times X^3 \to \mathbb{R}^+$ are given by

$$D_1(Y,V) = \frac{d(x,u) + d(y,v) + d(z,w)}{3}, \quad Y = (x,y,z), \quad V = (u,v,w) \in X^3$$

$$D_2(Y,V) = \max\{d(x,u), d(y,v), d(z,w)\}, \quad Y = (x,y,z), \quad V = (u,v,w) \in X^3,$$

then $\left(X^3,\sqsubseteq,D_1\right)$ and $\left(X^3,\sqsubseteq,D_2\right)$ are ordered metric spaces. The spaces $\left(X^3,D_1\right)$ and $\left(X^3,D_2\right)$ are complete if and only if (X,d) is a complete. Also, the spaces $\left(X^3,D_1,\sqsubseteq\right)$ and $\left(X^3,D_2,\sqsubseteq\right)$ are regular if and only if $(X,d,\preceq)$ is a such.

(2) *If $F : X^3 \to X$ and $g : X \to X$ and if F has the mixed g-monotone property, then the mapping $T_F : X^3 \to X^3$ given by*

$$T_F(Y) = (F(x,y,z), F(y,x,y), F(z,y,x)), \quad Y = (x,y,z) \in X^3$$

is T_g–non-decreasing with respect to $\sqsubseteq$, that is,

$$T_g(Y) \sqsubseteq T_g(V) \Rightarrow T_F(Y) \sqsubseteq T_F(V),$$

where $T_g(Y) = T_g(x,y,z) = (gx,gy,gz)$.

(3) *The mappings F and g are continuous (resp. compatible) if and only if T_F and T_g are continuous (resp. compatible).*
(4) *$F\left(X^3\right)$ (resp. $g\left(X^3\right)$) is complete in the metric spaces (X,d) if and only if $T_F\left(X^3\right)$ (resp. $T_g\left(X^3\right)$) is complete in the both $\left(X^3,D_1\right)$ and $\left(X^3,D_2\right)$.*
(5) *Mappings $F : X^3 \to X$ and $g : X \to X$ have a tripled coincidence point if and only if mappings T_F and T_g have a coincidence point in X^3.*
(6) *Mappings $F : X^3 \to X$ and $g : X \to X$ have a tripled common fixed point if and only if mappings T_F and T_g have a common fixed point in X^3.*

Lemma 1.5 [16]. *Let (X,d) be a metric space and let $\{y_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence in (X,d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences tend to ε^+ when $k \to \infty$: