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On a generalized core inverse

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ABSTRACT

The paper introduces the concept of a generalized core inverse of a matrix, which extends the notion of the core inverse defined by Baksalary and Trenkler [1]. While the original core inverse is restricted to matrices of index one, the generalized core inverse exists for any square matrix. Several properties of the new concept are identified with the derivations based essentially on partitioned representations of matrices. Some of the features of the generalized core inverse coincide with those attributed to the core inverse, but there are also such which characterize the core inverse only and not its generalization.

1. Preliminaries

Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ complex matrices. The symbols $\mathbf{A}^*, \mathcal{R}(\mathbf{A})$, and $\text{rk}(\mathbf{A})$ will denote the conjugate transpose, range (column space), and rank, respectively, of $\mathbf{A} \in \mathbb{C}_{m,n}$. Moreover, $\mathbf{I}_n$ will be the identity matrix of order n.

Besides the generalized core inverse, which will be introduced at the end of the present section, four types of generalized inverses of matrices will be involved in what follows. By $\mathbf{A}^\dagger \in \mathbb{C}_{n,m}$ we will mean the Moore–Penrose inverse of $\mathbf{A} \in \mathbb{C}_{m,n}$, i.e., the unique matrix satisfying the equations

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger, \quad \mathbf{A}\mathbf{A}^\dagger = (\mathbf{A}\mathbf{A}^\dagger)^*, \quad \mathbf{A}^\dagger\mathbf{A} = (\mathbf{A}^\dagger\mathbf{A})^*.$$

An important feature of the Moore–Penrose inverse is that it can be used to represent orthogonal projectors. For instance, $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger$ and $\mathbf{P}_{\mathbf{A}^*} = \mathbf{A}^\dagger\mathbf{A}$ are the orthogonal projectors onto $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A}^*)$, respectively.

Two further sets of matrix inverses being of interest are known in the literature as inner and outer generalized inverses, and for $\mathbf{A} \in \mathbb{C}_{m,n}$ are understood as

$$\mathbf{A}\{1\} = \{\mathbf{A}^- \in \mathbb{C}_{n,m} : \mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}\}$$

and

$$\mathbf{A}\{2\} = \{\mathbf{A}^= \in \mathbb{C}_{n,m} : \mathbf{A}^=\mathbf{A}\mathbf{A}^= = \mathbf{A}^=\},$$

respectively.

The fourth inverse is the core inverse introduced in [1]. Existence of such an inverse is restricted to square matrices only, and for a given $\mathbf{A} \in \mathbb{C}_{n,n}$ it is the unique matrix $\mathbf{A}^{\oplus} \in \mathbb{C}_{n,n}$ defined by the conditions

$$\mathbf{A}\mathbf{A}^{\oplus} = \mathbf{P}_\mathbf{A} \quad \text{and} \quad \mathcal{R}(\mathbf{A}^{\oplus}) \subseteq \mathcal{R}(\mathbf{A}).$$

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It is known that not every square matrix has the core inverse; a necessary and sufficient condition for a given matrix $\mathbf{A}$ to have such an inverse is that it is of index one, i.e., $\mathrm{rk}(\mathbf{A}^2) = \mathrm{rk}(\mathbf{A})$. The matrices of index one are called group matrices [2, p. 242] or core matrices [3, p. 47].

Additional symbols used in the present paper are $\mathbb{C}_n^{\mathsf{P}}, \mathbb{C}_n^{\mathsf{OP}}$, and $\mathbb{C}_n^{\mathsf{EP}}, \mathbb{C}_n^{\mathsf{NM}}, \mathbb{C}_n^{\mathsf{CM}}$ and denote the sets of oblique projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices), and EP (range-Hermitian), normal, core (index one or group) matrices, respectively, i.e.,

$$\mathbb{C}_n^{\mathsf{P}} = \{\mathbf{A} \in \mathbb{C}_{n,n} : \mathbf{A}^2 = \mathbf{A}\},$$

$$\mathbb{C}_n^{\mathsf{OP}} = \{\mathbf{A} \in \mathbb{C}_{n,n} : \mathbf{A}^2 = \mathbf{A} = \mathbf{A}^*\},$$

$$\mathbb{C}_n^{\mathsf{EP}} = \{\mathbf{A} \in \mathbb{C}_{n,n} : \mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}\},$$

$$\mathbb{C}_n^{\mathsf{NM}} = \{\mathbf{A} \in \mathbb{C}_{n,n} : \mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}\},$$

$$\mathbb{C}_n^{\mathsf{CM}} = \{\mathbf{A} \in \mathbb{C}_{n,n} : \mathrm{rk}(\mathbf{A}^2) = \mathrm{rk}(\mathbf{A})\}.$$

Recall that every matrix $\mathbf{A} \in \mathbb{C}_{n,n}$ of rank r can be represented in the form

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma K} & \mathbf{\Sigma L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{1.1}$$

where $\mathbf{U} \in \mathbb{C}_{n,n}$ is unitary, $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1\mathbf{I}_{r_1}, \ldots, \sigma_t\mathbf{I}_{r_t})$ is the diagonal matrix of singular values of $\mathbf{A}, \sigma_1 > \sigma_2 > \cdots > \sigma_t > 0, r_1 + r_2 + \cdots + r_t = r$, and $\mathbf{K} \in \mathbb{C}_{r,r}, \mathbf{L} \in \mathbb{C}_{r,n-r}$ satisfy

$$\mathbf{K}\mathbf{K}^* + \mathbf{L}\mathbf{L}^* = \mathbf{I}_r; \tag{1.2}$$

see [2, Corollary 6].

The lemma below, which will be useful in the subsequent considerations, is quoted here after [4, Section 1].

Lemma 1. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of rank r and have representation* (1.1). *Then:*

(i) $\mathbf{A} \in \mathbb{C}_n^{\mathsf{P}}$ *if and only if* $\mathbf{\Sigma K} = \mathbf{I}_r$,
(ii) $\mathbf{A} \in \mathbb{C}_n^{\mathsf{OP}}$ *if and only if* $\mathbf{L} = \mathbf{0}$, $\mathbf{\Sigma} = \mathbf{I}_r$, $\mathbf{K} = \mathbf{I}_r$,
(iii) $\mathbf{A} \in \mathbb{C}_n^{\mathsf{EP}}$ *if and only if* $\mathbf{L} = \mathbf{0}$,
(iv) $\mathbf{A} \in \mathbb{C}_n^{\mathsf{NM}}$ *if and only if* $\mathbf{L} = \mathbf{0}$, $\mathbf{K\Sigma} = \mathbf{\Sigma K}$,
(v) $\mathbf{A} \in \mathbb{C}_n^{\mathsf{CM}}$ *if and only if* $\mathrm{rk}(\mathbf{K}) = r$.

If $\mathbf{A}$ is of the form (1.1), then

$$\mathbf{A}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \tag{1.3}$$

and

$$\mathbf{A}^{\oplus} = \mathbf{U}\begin{pmatrix} (\mathbf{\Sigma K})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*; \tag{1.4}$$

see [1]. Note that $\mathbf{A}^{\oplus}$ merely exists when $\mathbf{K}$ is nonsingular, which by Lemma 1(v) means that $\mathbf{A} \in \mathbb{C}_n^{\mathsf{CM}}$.

A crucial role in the present paper is played by the notion of the generalized core inverse, which is understood according to the following definition.

Definition 1. Let $\mathbf{A} \in \mathbb{C}_{n,n}$. A matrix $\mathbf{A}^\diamondsuit \in \mathbb{C}_{n,n}$ satisfying

$$\mathbf{A}^\diamondsuit = (\mathbf{A}\mathbf{P}_{\mathbf{A}})^\dagger \tag{1.5}$$

is called the generalized core inverse of $\mathbf{A}$.

It is seen from (1.5) that $\mathbf{A}^\diamondsuit$ exists for every $\mathbf{A} \in \mathbb{C}_{n,n}$ and is unique.

In the next section various properties of the generalized core inverse are identified, with emphasis laid on those characterizations which demonstrate connections of the inverse with various known classes of matrices. It will be shown that some of the features of the generalized core inverse coincide with those attributed to the core inverse, but there are also such which characterize the core inverse only and not its generalization. It is noteworthy that $\mathbf{A}^\diamondsuit$ is not the only generalization of the core inverse known in the literature, for another notion of the kind was introduced and investigated in [5].