Applied Mathematics and Computation 236 (2014) 458–470

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Newton method in the context of quaternion analysis

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ARTICLE INFO

Keywords:
Newton method
Quaternions
Radial derivative

ABSTRACT

In this paper we propose a version of Newton method for finding zeros of a quaternion function of a quaternion variable, based on the concept of quaternion radial derivative. Several numerical examples involving elementary functions are presented.

1. Introduction

Since 1928 Fueter, one of the founders of quaternion analysis [1,2], tried to develop a function theory to generalize the theory of holomorphic functions of one complex variable, by considering quaternion valued functions of a quaternion variable. Nowadays, this well known and developed theory is recognized as a powerful tool for modeling and solving problems in both theoretical and applied mathematics. For a survey on quaternion analysis and a list of references we refer to the book [3]; a historical perspective of the subject and several applications can be found in [4].

In this work we revisit the classical Newton method for finding roots (or zeros) of a complex function and propose a quaternion analogue, based on the concept and properties of quaternion radially holomorphic functions. We show that for a certain class of functions (including simple polynomials and other elementary functions) this method produces the same sequence as the classical Newton method for vector valued functions. In this way, we can obtain, with less computational effort, local quadratic convergence for a class of quaternion functions.

This idea was already considered by Janovská and Opfer in [5], where the authors formally adapted, for the first time, Newton method for finding roots of Hamilton quaternions, by considering the quaternion equation $x^n - a = 0$. More recently, Kalantari in [6], using algebraic-combinatorial arguments, proposed a Newton method for finding roots of special quaternion polynomials. Working in the framework of quaternion analysis, we can provide a motivation for the techniques used in those works and simultaneously extend the applicability of the method.

The paper is organized as follows. In Section 2 we introduce the basic notations and the results that are needed for our work in Section 3.

Section 3 contains the main results of the paper. Here, by making use of the theory given in Section 2, and after establishing new properties on the radial derivative of a special class of functions, we propose a Newton method in the framework of quaternion analysis.

Finally, in Section 4 several numerical examples illustrating the applicability of the aforementioned methods are presented.

2. Quaternion analysis

We start by first recalling some basic results concerning Hamilton quaternion algebra $\mathbb{H}$, which can be found in classic books on this subject. For results concerning quaternion analysis we refer to [7,8,3].

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http://dx.doi.org/10.1016/j.amc.2014.03.050
0096-3003/

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

This non-commutative product generates the well known algebra of real quaternions $\mathbb{H}$. The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying the element $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ (or the column vector in $\mathbb{R}^{4\times1}, x = (x_0\ x_1\ x_2\ x_3)^T$) with the element $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$. Throughout this paper, we will also use the symbol $\boldsymbol{x}$ to represent an element in $\mathbb{R}^4$, whenever we need to distinguish the structure of $\mathbb{H}$ from $\mathbb{R}^4$.

The *real* or *scalar part* of a quaternion $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is denoted by $Sc\, x$ and is equal to x_0, the *vector part* of x is $\underline{x} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and therefore x can be written as $x = x_0 + \underline{x}$. The *conjugate* of x is $\bar{x} := x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k} = x_0 - \underline{x}$. The mapping $x \mapsto \bar{x}$ is called *conjugation* and has the property $\overline{xy} = \bar{y}\,\bar{x}$, for all $x, y \in \mathbb{H}$. The norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x$ and coincides with the corresponding Euclidean norm of $\boldsymbol{x}$, as a vector in $\mathbb{R}^4$. It follows that $|xy| = |x||y|$ and each non-zero $x \in \mathbb{H}$ has an inverse given by $x^{-1} = \frac{\bar{x}}{|x|^2}$. Moreover, $|x|^{-1} = |x^{-1}|$.

In this work we are going to consider also the representation of the quaternion $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ by means of the real matrix in $\mathbb{R}^{4\times4}$

$$L_x := \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}. \tag{1}$$

This representation is called *matrix left representation* of x and can be associated with the product of quaternions

$$\begin{aligned} xy &= (x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})(y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}) = (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3)\mathbf{i} \\ &\quad + (x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3)\mathbf{j} + (x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3)\mathbf{k}, \end{aligned} \tag{2}$$

through the identification

$$z = xy \ \rightarrow \ \boldsymbol{z} = L_x\boldsymbol{y}, \tag{3}$$

where $\boldsymbol{y}$ is the (column) vector in $\mathbb{R}^4$ corresponding to the quaternion y.

Any arbitrary nonreal quaternion x can also be written in the so-called *complex-like* form

$$x = x_0 + \omega(\underline{x})|\underline{x}|, \tag{4}$$

where

$$\omega(\underline{x}) := \frac{\underline{x}}{|\underline{x}|}$$

belongs to the unit sphere in $\mathbb{R}^3$. Since $\omega(\bar{\underline{x}}) = \overline{\omega(\underline{x})} = -\omega(\underline{x})$, it follows immediately that $\omega(\underline{x})^2 = -\omega(\underline{x})\overline{\omega(\underline{x})} = -|\omega(\underline{x})|^2 = -1$. In other words, we can consider that ω behaves like the imaginary unit and therefore the complex-like form (4) is similar to the complex form $a + ib$. We use the convention $\omega(\underline{x}) := 0$, for real quaternions x. The following properties play an important role in the present work.

Proposition 1. *If $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ are quaternions, the following statements are equivalent:*

(i) $xy = yx$;
(ii) $x_1y_2 = x_2y_1$ *and* $x_1y_3 = x_3y_1$ *and* $x_2y_3 = x_3y_2$;
(iii) $\omega(\underline{x})\omega(\underline{y}) = \omega(\underline{y})\omega(\underline{x})$;
(iv) $\omega(\underline{x}) = \pm\omega(\underline{y})$.

Proof. The equivalence between the first three statements follows at once from the multiplication rules (cf. (2)). If $\omega(\underline{x}) = \pm\omega(\underline{y})$, clearly $\omega(\underline{x})\omega(\underline{y}) = \mp1 = \omega(\underline{y})\omega(\underline{x})$ and we conclude that (iv) implies (iii).

Now, we prove that (ii) implies (iv) for the case of nonreal quaternions. If y is nonreal, we can assume that, for example, $y_1 \neq 0$. Using (ii) one can write $x_2 = x_1\frac{y_2}{y_1}$ and $x_3 = x_1\frac{y_3}{y_1}$ and this, in turn, implies that $\underline{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = (y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})\frac{x_1}{y_1}$. This means that $|\underline{x}| = |\underline{y}|\frac{|x_1|}{|y_1|}$ and the result $\omega(\underline{x}) = \frac{\underline{x}}{|\underline{x}|} = \frac{\underline{y}}{|\underline{y}|}\frac{x_1|y_1|}{|x_1|y_1} = \pm\omega(\underline{y})$ follows. $\square$

Corollary 1. *If x and y are quaternions such that $\omega(\underline{x})$ and $\omega(\underline{y})$ commute then:*

(i) $x\bar{y} = \bar{y}x$;
(ii) $xy^{-1} = y^{-1}x$.