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More on a rational second-order difference equation

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ABSTRACT

Here we give natural explanations for some recent relations for the solutions of the following difference equation

$$z_{n+2}=\frac{z_{n+1}^3}{z_n^2-2z_{n+1}^2},\quad n\in\mathbb{N}_0,$$

and extend them for the case of complex initial values. We also present some other properties related to solutions of the equation, as well as some properties of the solutions of an associate difference equation of first order.

1. Introduction

Nonlinear difference equations of various types have attracted considerable attention in last few decades (see, for example, [1–31] and the references therein). Recently there has been some renewed interest in solving difference equations or in finding some relationships and/or invariants which are satisfied by their solutions and which can be used in studying of their long-term behavior (see, for example, [1,3–5,12–14,17–23,25–31]).

In note [5], De Bruyn considered the following difference equation

$$z_{n+2}=\frac{z_{n+1}^3}{z_n^2-2z_{n+1}^2},\quad n\in\mathbb{N}_0,\tag{1}$$

where the initial values z_0 and z_1 are real numbers satisfying some additional conditions.

He proved the following two relations for solutions of Eq. (1). The first one looks like a "law of conservation of energy" for Eq. (1)

$$y_n=z_0+z_1+\cdots+z_n+\frac{z_n^2}{2z_{n+1}}=z_0+\frac{z_0^2}{2z_1},\quad n\in\mathbb{N}_0,\tag{2}$$

while the second one is a reduction of Eq. (1) to the first order difference equation

$$z_{n+1}=\frac{z_1z_n^2}{\sqrt{z_0^2(z_0^2-4z_1^2)+4z_1^2z_n^2}},\quad n\in\mathbb{N}_0.\tag{3}$$

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Relation (2) was proved by calculating the difference $y_n - y_{n-1}$, while relation (3) was proved by the method of induction, so that both relations are essentially not explained theoretically. Relations (2) and (3) are used in calculating the sum $\sum_{n=0}^{\infty} z_n$, under some conditions posed on initial values z_0 and z_1.

Motivated by [5], here we give some natural explanations for relations (2) and (3), extend them for the case of complex initial values z_0 and z_1, and present some other properties of solutions of Eq. (1). We also present some properties of solutions of an associate first order difference equation to Eq. (1) (see Eq. (7)). Eq. (7) is folklore and highly investigated (see, for example, [2,10] and the references therein) so we expect that majority of the facts that we mention here related to solutions of Eq. (7) could be known to the experts in the research field, but we will present them for the completeness and the benefit of the reader.

2. On an associate first order difference equation to Eq. (1)

Eq. (1) is a particular case of the following difference equation of second order

$$z_{n+2} = z_{n+1} f\left(\frac{z_{n+1}}{z_n}\right), \quad n \in \mathbb{N}_0, \tag{4}$$

where $z_0, z_1 \in \mathbb{C}$.

Eq. (4) is reduced to the next difference equation

$$u_{n+1} = f(u_n), \quad n \in \mathbb{N}_0,$$

of the first order, by using the following natural change of variables

$$u_n = \frac{z_{n+1}}{z_n}, \quad n \in \mathbb{N}_0.$$

Now note that if $z_1 = 0$, then from (1) we get that $z_2 = 0$, from which it would follow that z_3 is not defined. This is an obvious reason why De Bruyn in studying Eq. (1) posed the assumption $z_1 \neq 0$.

Note also that if $z_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then from (1) it follows that $z_{n_0-1} = 0$, so repeating the procedure we finally get $z_1 = 0$, but as we have already showed such a solution is not defined. Hence, from now on we may consider only the solutions such that $z_n \neq 0$ for every $n \in \mathbb{N}$.

For the case when $z_0, z_1 \in \mathbb{R} \setminus \{0\}$, De Bruyn also posed the following condition

$$|z_0| \geqslant 2|z_1|, \tag{5}$$

which is a sufficient condition guaranteeing that a solution of Eq. (1) is well-defined, that is, every term of the solution $(z_n)_{n\in\mathbb{N}_0}$ of Eq. (1) with initial values z_0 and z_1 satisfying (5) is defined and has a finite value (see, Lemma 1 in [5]).

Note that for the case of Eq. (1) it is more suitable to use the following change of variables

$$w_n = \frac{z_n}{z_{n+1}}, \quad n \in \mathbb{N}_0. \tag{6}$$

Namely, by using the change of variables (6), Eq. (1) is transformed into the following well-known difference equation

$$w_{n+1} = w_n^2 - 2, \quad n \in \mathbb{N}_0. \tag{7}$$

As we have mentioned, Eq. (7) is folklore so numerous properties of their solutions are known to the experts (for the case of real-valued solutions of Eq. (7), see, e.g., [10], while for the complex-valued see, e.g., [2]). Nevertheless, we will present here some of the properties and relate them to the properties of solutions of Eq. (1).

Assume that condition (5) holds. Then since we assume that $z_1 \neq 0$, the condition obviously can be written in the following form

$$|w_0| \geqslant 2. \tag{8}$$

Assume that $|w_k| \geqslant 2$, for some $k \in \mathbb{N}_0$, then from (7) and the triangle inequality, it follows that

$$|w_{k+1}| = |w_k^2 - 2| \geqslant |w_k|^2 - 2 \geqslant 2.$$

Hence, by induction we have that condition (8) implies

$$|w_n| \geqslant 2, \quad n \in \mathbb{N}_0. \tag{9}$$

Note that if the inequality in (8) is strict, that is, if

$$|w_0| > 2, \tag{10}$$

then the inequality in (9) is also strict for every $n \in \mathbb{N}_0$.

Using (9) and the triangle inequality, we further have that

$$|w_{n+1}| - |w_n| = |w_n^2 - 2| - |w_n| \geqslant |w_n|^2 - |w_n| - 2 = (|w_n| - 2)(|w_n| + 1) \geqslant 0, \tag{11}$$