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Eventual periodicity of some systems of max-type difference equations

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ABSTRACT

The periodicity of the next system of max-type difference equations

$$x_n^{(1)} = \max_{1\leqslant i_1\leqslant m_1}\left\{f_{1i_1}\left(x^{(1)}_{n-k^{(1)}_{i_1,1}}, x^{(2)}_{n-k^{(1)}_{i_1,2}}, \ldots, x^{(l)}_{n-k^{(1)}_{i_1,l}}, n\right), x^{(\sigma(1))}_{n-t_1 s}\right\},$$
$$x_n^{(2)} = \max_{1\leqslant i_2\leqslant m_2}\left\{f_{2i_2}\left(x^{(1)}_{n-k^{(2)}_{i_2,1}}, x^{(2)}_{n-k^{(2)}_{i_2,2}}, \ldots, x^{(l)}_{n-k^{(2)}_{i_2,l}}, n\right), x^{(\sigma(2))}_{n-t_2 s}\right\},$$
$$\vdots$$
$$x_n^{(l)} = \max_{1\leqslant i_l\leqslant m_l}\left\{f_{li_l}\left(x^{(1)}_{n-k^{(l)}_{i_l,1}}, x^{(2)}_{n-k^{(l)}_{i_l,2}}, \ldots, x^{(l)}_{n-k^{(l)}_{i_l,l}}, n\right), x^{(\sigma(l))}_{n-t_l s}\right\},$$

$n \in \mathbb{N}_0$, where $s, l, m_j, t_j, k^{(j)}_{i_j,h} \in \mathbb{N}$, $j, h \in \{1, \ldots, l\}$, $(\sigma(1), \sigma(2), \ldots, \sigma(l))$ is a permutation of $(1, 2, \ldots, l)$, and $f_{ji_j} : (0,\infty)^l \times \mathbb{N}_0 \to (0,\infty)$, $j \in \{1, \ldots, l\}$, $i_j \in \{1, \ldots, m_j\}$, is studied. It is shown that under some conditions posed on functions f_{ji_j} all positive solutions of the system are eventually periodic with period Ts, for some $T \in \mathbb{N}$.

1. Introduction

The investigation of nonlinear difference equations and systems which are not analogs of differential ones has attracted some recent attention (see, e.g., [1–45]). One of the classes of such equations/systems are max-type difference equations/systems (see, e.g., [1,5,8–12,17,18,25–31,34,36,39,40,43–45] and the related references therein). While majority of papers in the area are devoted to some max-type difference equations, only few of them deal with systems of max-type difference equations (see [16–18,36,39,40]), notwithstanding the fact that there are a lot of papers on various types of systems of difference equations (see, e.g., [6,13–20,32,35–42]).

A vector sequence $\vec{x}_n = (x_n^{(1)}, \ldots, x_n^{(l)}), n \geqslant -k$, is called eventually periodic with period $p \in \mathbb{N}$ if there is an $n_0 \geqslant -k$, such that

$$x^{(j)}_{n+p} = x^{(j)}_n, \quad \text{for } n \geqslant n_0$$

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for every $j \in \{1, \ldots, l\}$. If $n_0 = -k$, then we say that the vector sequence $(\vec{x}_n)_{n=-k}^{\infty}$ is periodic with period p. Period p is prime if there is no $\hat{p} \in \mathbb{N}$, $\hat{p} < p$ which is a period for the vector sequence. For $p = 1$ the sequences are called eventually constant or trivial (see, e.g., [7,24]). For $l = 1$ we obtain standard notions of eventual periodicity and periodicity for scalar sequences. For some results in the area, see, e.g., [1,4–6,8–12,23,30,33, 34,36,44], and the related references therein.

Solutions of max-type difference equations and systems, are quite often, eventually periodic or periodic. This seems connected with the fact that maxima operators are not smooth.

As one of the basic max-type difference equations special cases of the next equation

$$x_n = \max\left\{\frac{\alpha_n}{x_{n-k}}, x_{n-s}\right\}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $k, s \in \mathbb{N}$, and $(\alpha_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$, have been studied for some time, especially for some concrete values of k and s and for the case when α_n is constant or periodic (see, e.g., [5,30] and the related references therein).

The periodicity character of positive solutions of an extension of Eq.(1) was explained in [34]. Results in [34] was later extended to some systems of max-type difference equations in [36]. However, the periodicity character of positive solutions of some related max-type systems of difference equations, such as is the following one

$$x_n = \max\left\{\frac{A_n}{x_{n-k}}, y_{n-s}\right\}, \quad y_n = \max\left\{\frac{B_n}{y_{n-k}}, z_{n-s}\right\}, \quad z_n = \max\left\{\frac{C_n}{z_{n-k}}, x_{n-s}\right\}, \tag{2}$$

where $k, s \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}, (B_n)_{n\in\mathbb{N}_0}$ and $(C_n)_{n\in\mathbb{N}_0}$ are periodic sequences of real numbers, was not explained in [36].

Here we prove a general result which includes all the known results in the literature as well as system of difference equations (2) for the case when $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and $(C_n)_{n\in\mathbb{N}_0}$ are periodic sequences of positive numbers. We prove also a result which deals with a related system of difference equations with minima operators.

2. Main results

In this section we formulate and prove the main results in this paper. Our first result deals with a max-type system which includes into itself also system of difference equations (2). Before this recall that $\operatorname{lcm}(y_1, \ldots, y_h)$ denotes the least common multiple of natural numbers $y_1, \ldots, y_h$.

Theorem 1. *Consider the next system of difference equations*

$$\begin{aligned}
x_n^{(1)} &= \max_{1\leqslant i_1\leqslant m_1}\left\{f_{1i_1}\left(x^{(1)}_{n-k^{(1)}_{i_1,1}}, x^{(2)}_{n-k^{(1)}_{i_1,2}}, \ldots, x^{(l)}_{n-k^{(1)}_{i_1,l}}, n\right), x^{(\sigma(1))}_{n-t_1 s}\right\},\\
x_n^{(2)} &= \max_{1\leqslant i_2\leqslant m_2}\left\{f_{2i_2}\left(x^{(1)}_{n-k^{(2)}_{i_2,1}}, x^{(2)}_{n-k^{(2)}_{i_2,2}}, \ldots, x^{(l)}_{n-k^{(2)}_{i_2,l}}, n\right), x^{(\sigma(2))}_{n-t_2 s}\right\},\\
&\ \ \vdots\\
x_n^{(l)} &= \max_{1\leqslant i_l\leqslant m_l}\left\{f_{li_l}\left(x^{(1)}_{n-k^{(l)}_{i_l,1}}, x^{(2)}_{n-k^{(l)}_{i_l,2}}, \ldots, x^{(l)}_{n-k^{(l)}_{i_l,l}}, n\right), x^{(\sigma(l))}_{n-t_l s}\right\},
\end{aligned} \tag{3}$$

$n \in \mathbb{N}_0$, where $s, l, m_j, t_j, k^{(j)}_{i_j,h} \in \mathbb{N}$, $j, h \in \{1, \ldots, l\}$, $(\sigma(1), \sigma(2), \ldots, \sigma(l))$ is a permutation of $(1, 2, \ldots, l)$, and where

(a) *the functions $f_{ji_j} : (0,\infty)^l \times \mathbb{N}_0 \to (0,\infty)$, for $j \in \{1, \ldots, l\}$ and $i_j \in \{1, \ldots, m_j\}$, are nonincreasing in all arguments; or*
(b) *the functions $f_{ji_j} : (0,\infty)^l \times \mathbb{N}_0 \to (0,\infty)$, for $j \in \{1, \ldots, l\}$ and $i_j \in \{1, \ldots, m_j\}$, are nonincreasing in the first l arguments and*

$$f_{ji_j}(a_1, \ldots, a_l, b) = f_{ji_j}(a_1, \ldots, a_l, b + \tau_{ji_j}) \tag{4}$$

for every $(a_1, \ldots, a_l) \in (0,\infty)^l, b \in \mathbb{N}_0$ and for some $\tau_{ji_j} \in \mathbb{N}, j \in \{1, \ldots, l\}$ and $i_j \in \{1, \ldots, m_j\}$.

Then every positive solution to system (3) is eventually periodic with (not necessarily prime) period Ts, where

$$T := \operatorname{lcm}\left(\tau\sum_{j=1}^{p(1)} t_{\sigma^{[j-1]}(1)}, \ldots, \tau\sum_{j=1}^{p(l)} t_{\sigma^{[j-1]}(l)}\right), \tag{5}$$

$$\tau := \operatorname{lcm}_{1\leqslant j\leqslant l, 1\leqslant i_j\leqslant m_j} \tau_{ji_j} \tag{6}$$

and $p(j), j = \overline{1,l}$, *are the least natural numbers such that $\sigma^{[p(j)]}(j) = j$, $j = \overline{1,l}$.*